UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C331: Algebra I

COURSE CODE : **MATHC331**

UNIT VALUE : **0.50**

DATE : **04-MAY-04**

TIME : **14.30**

TIME ALLOWED : **2 Hours**

04-C0921-3-30

TURN OVER

All *questions may be attempted but only marks obtained on the best* **four** *solutions will count. The use of an electronic calculator is* **not** *permitted in this examination. Throughout all rings are commutative with a 1 and all free modules are assumed to be of finite rank.*

1. Let R be an integral domain.

   (a) For $a, b \in R$, state what it means to say that a is associated to b. Show this happens if and only if $a \,|\, b$ and $b \,|\, a$.

   (b) For $a \in R$ state what it means to say that a is (i) an atom, (ii) a prime. If a is a prime, show that a is an atom.

   (c) State what it means to say that R is (i) a Euclidean domain, (ii) a principal ideal domain, (iii) a unique factorization domain. Show that any Euclidean domain is a principal ideal domain.

   (d) Find the atomic factorization of $7 + i \in \mathbb{Z}[i]$. Justify your answer.

   [You may assume any standard results about $\mathbb{Z}[i]$ which you require.]

2. (a) State, *without proof*, the first isomorphism theorem for modules. Let P be a submodule of a module M. Show that the submodules K of M/P are of the form $K = Q/P$, where $P \subseteq Q \leqslant M$.

   (b) State what it means to say that a R-module M is cyclic. Show that M is cyclic if and only if $M \cong R/I$ for some ideal I of R.

   (c) Let R be an integral domain. Show that every submodule of every cyclic R-module is cyclic if and only if R is a principal ideal domain.

3. Let (R, N) be a Euclidean Domain and $A \in {}^mR^n$.

Define the Smith Normal Form of A and state to what extent it is unique. Describe, *without proof*, how to reduce A to Smith Normal Form.

Let

$$A = \begin{bmatrix} 6 & 4 & 8 \\ 4 & 4 & -4 \\ 8 & 8 & 4 \\ 2 & 4 & -16 \end{bmatrix} \in {}^4\mathbb{Z}^3.$$

Find $P \in GL(4, \mathbb{Z})$ and $Q \in GL(3, \mathbb{Z})$ such that PAQ is in Smith Normal Form.

4. (a) State, *without proof*, the classification of finitely generated abelian groups. To what extent is the classification unique?

   Let A be a finitely generated abelian group. Show that A is a torsion group if and only if A is finite.

   (b) Classify all abelian groups A of order 2025 by means of both invariant factors and elementary divisors.

      (i) Find all such distinct A satisfying $135A = 0$.

      (ii) Find all such distinct A such that $A = B \oplus C$ where B and C are non-zero cyclic groups.

5. (a) Let $\alpha : V \longrightarrow V$ be a linear map, where V is a finite-dimensional vector space over a field $\mathbb{F}$. Explain how V can be considered as a finitely generated torsion module over $\mathbb{F}[x]$.

   (b) Let $A \in {}^n\mathbb{C}^n$. *Briefly* explain, *without proof,* how the structure theory for finitely generated torsion modules over a principal ideal domain can be used to classify A under similarity. In particular, define the Rational Canonical Form and Jordan Normal Form of A and explain their relation to this classification.

   (c) Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \in {}^4\mathbb{C}^4.$$

   By reducing the characteristic matrix of A appropriately, find the Rational Canonical Form and Jordan Normal Form of A for **all** $\lambda \in \mathbb{C}$.

END OF PAPER