UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C331: Algebra I

COURSE CODE : **MATHC331**

UNIT VALUE : **0.50**

DATE : **02-MAY-03**

TIME : **14.30**

TIME ALLOWED : **2 Hours**

03-C0913-3-40

TURN OVER

All *questions may be attempted but only marks obtained on the best* **four** *solutions will count. The use of an electronic calculator is* **not** *permitted in this examination.*

Throughout all rings are assumed to be commutative with a 1 and all free modules to be of finite rank.

1. (a) Let R be an integral domain. Define what it means to say that

   (i) $a \in R$ is an atom,

   (ii) R satisfies the ascending chain condition on principal ideals.

   If R satisfies the ascending chain condition on principal ideals, show that every non-zero, non-unit element of R can be written as the product of atoms.

   (b) Let $m \in \mathbb{Z}$ not be a square in $\mathbb{Z}$. Define $(\mathbb{Z}[\sqrt{m}], N)$ in the usual way with $\mathbb{Z}[\sqrt{m}] = \{a + b\sqrt{m} : a, b \in \mathbb{Z}\} \leqslant \mathbb{C}$ and $N(a + b\sqrt{m}) = |a^2 - mb^2|$. Outline briefly, **without proof**, why $\mathbb{Z}[\sqrt{m}]$ satisfies the ascending chain condition on principal ideals. Write 6 as a product of atoms in **both** $\mathbb{Z}[i]$ and $\mathbb{Z}[\sqrt{-2}]$ **justifying** your answers.
   [For both parts of (b) you may assume any of the standard properties of $(\mathbb{Z}[\sqrt{m}], N)$.]

2. (a) Define what it means to say that $\alpha : M \longrightarrow N$ is an R-homomorphism for R-modules M and N.

   If this is the case, define $\mathrm{Im}(\alpha)$ and $\mathrm{Ker}(\alpha)$ and show that

   (i) $\mathrm{Im}(\alpha) \leqslant N$,

   (ii) $\mathrm{Ker}(\alpha) \leqslant M$,

   (iii) $\mathrm{Im}(\alpha) \cong M/\mathrm{Ker}(\alpha)$.

   (b) Suppose that A and B are submodules of an R-module M.

   (i) Show that $(A + B)/B \cong A/(A \cap B)$.

   (ii) If further $A + B = M$, show that $M/(A \cap B) \cong M/A \oplus M/B$.

3. Let (R, N) be a Euclidean domain and $A \in {}^mR^n$. Define the Smith Normal Form of A and state to what extent it is unique. Describe how to reduce A to Smith Normal Form. Briefly explain, **without proof**, why the Smith Normal Form is unique.

   Find the Smith Normal Form of $A \in {}^3\mathbb{Z}^4$ where

$$A = \begin{bmatrix} 0 & 6 & 6 & 6 \\ 2 & 8 & 2 & 4 \\ -2 & -5 & 2 & -2 \end{bmatrix}.$$

4. (i) Let M be a module over an integral domain R. State what it means to say that $m \in M$ is a torsion element of M. Show that the set $T(M)$ of all torsion elements of M forms a submodule of M. What does it mean to say that M is a torsion module?

   (ii) Describe, **without proof**, the classification of finitely generated torsion modules over principal ideal domains by means of invariant factors and elementary divisors. State to what extent the classifications are unique.

   (iii) Using the above classification, show how to find all finite abelian groups A with $|A| = p^n$, where p is a prime.

   (iv) Find all non-isomorphic abelian groups of order 360 classifying them by means of invariant factors and elementary divisors.

5. (a) Let $\alpha : V \longrightarrow V$ be a linear map, where V is a finite-dimensional vector space over a field $\mathbb{F}$.

   (i) Explain how V can be considered as a finitely generated torsion module over $\mathbb{F}[x]$.

   (ii) Suppose that $\alpha : V \longrightarrow V$ gives rise to a cyclic $\mathbb{F}[x]$-module $V = \mathbb{F}[x]\underline{v} \cong \mathbb{F}[x]/(d)$ where $\underline{v} \in V$ and d is the minimum polynomial of α. Show, **with proof**, how to find a basis e for V such that $[\alpha]_e^e = C_d$ where C_d is the companion matrix of d.

   (b) Let

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in {}^4\mathbb{C}^4.$$

By reducing the characteristic matrix of A appropriately, find the Rational Canonical Form and Jordan Normal Form of A.

END OF PAPER