UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Sc. *M.Sci.*

Mathematics C331: Algebra I

COURSE CODE	:	**MATHC331**
UNIT VALUE	:	**0.50**
DATE	:	**10-MAY-02**
TIME	:	**14.30**
TIME ALLOWED	:	**2 hours**

02-C0911-3-40

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

Throughout all rings are commutative with a 1 and all free modules are assumed to be of finite rank.

1. Define the following terms for an integral domain:

   (i) atom, (ii) prime, (iii) Euclidean domain, (iv) principal ideal domain, (v) unique factorization domain.

   (a) If (R, N) is a Euclidean domain, show that R is a principal ideal domain.

   (b) Let $\mathbb{Z}[i]$ be the integral domain $\{a + bi : a, b \in \mathbb{Z}\}$. Define $N : \mathbb{Z}[i] \to \mathbb{N}$ by $a + bi \mapsto a^2 + b^2$. Show that $(\mathbb{Z}[i], N)$ is a Euclidean domain.

   (c) Find a factorization of $6 + 8i \in \mathbb{Z}[i]$ into atoms. Justify your answer. [You may assume any of the usual properties of $\mathbb{Z}[i]$.]

2. What does it mean to say that a module is (i) finitely generated, (ii) cyclic?

   (a) Suppose that P is a submodule of a module M. Show that

      (i) if M is finitely generated or cyclic, then so is M/P,

      (ii) if both P and M/P are finitely generated, then M is finitely generated.

   (b) Show that any submodule of a free module over a principal ideal domain is finitely generated.

3. Let (R, N) be a Euclidean domain and $A \in {}^mR^n$.

   Define the Smith Normal Form of A and state to what extent it is unique. Describe *without proof* how to reduce A to Smith Normal Form.

   Let $A = \begin{bmatrix} 4 & 12 & 8 \\ 4 & 20 & 14 \end{bmatrix} \in {}^2\mathbb{Z}^3$. Find $P \in GL(2, \mathbb{Z}), Q \in GL(3, \mathbb{Z})$ such that PAQ is in Smith Normal Form.

4. (a) State, *without proof*, the cl[illegible]ation of finitely generated abelian groups. To what [illegible]xtent is the classifica[illegible] unique?

   Let A be a finitely generated abelian group. Show that A is a torsion group if and only if A is finite.

   (b) Classify all abelian groups of order 72 by means of both invariant factors and elementary divisors.

   (c) Find the abelian group given by generators e_1, e_2, e_3, e_4, e_5 subject to the relations

$$\begin{array}{ccccccccccc} e_1 & + & e_2 & + & e_3 & + & 2e_4 & + & 2e_5 & = & 0 \\ e_1 & + & 3e_2 & + & e_3 & + & 4e_4 & + & 2e_5 & = & 0 \\ & & 6e_2 & + & 6e_3 & + & 12e_4 & + & 6e_5 & = & 0 \\ 2e_1 & + & 4e_2 & + & 2e_3 & + & 6e_4 & + & 4e_5 & = & 0 \end{array}$$

5. (a) Let $\alpha : V \to V$ be a linear map, where V is a finite-dimensional vector space over a field $\mathbb{F}$. Explain how V can be considered as a finitely generated torsion module over $\mathbb{F}[x]$.

   (b) Let $A \in {}^n\mathbb{C}^n$. *Briefly* explain, *without proof*, how the structure theory for finitely generated torsion modules over a principal ideal domain can be used to classify A under similarity. In particular, define the Rational Canonical Form and Jordan Normal Form of A and explain their relation to this classification.

   (c) Let $A = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \in {}^4\mathbb{C}^4$. By reducing the characteristic matrix of A appropriately, find the Rational Canonical Form and Jordan Normal Form of A.