UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M222: Algebra 4: Groups and Rings

COURSE CODE	:	**MATHM222**
UNIT VALUE	:	**0.50**
DATE	:	**11-MAY-06**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

06-C1046-3-120

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Let G be a finite group. Explain what is meant by the order, ord g , of $g \in G$.

   Define the *kernel* $\text{Ker}(\varphi)$ and *image* $\text{Im}(\varphi)$ of a group homomorphism $\varphi : G \to H$.

   State and prove a relationship which holds between $\text{Ker}(\varphi)$ and $\text{Im}(\varphi)$.

   Deduce that if $x \in G$ then ord $\varphi(x)$ divides both $|G|$ and $|H|$.

   Let $\varphi_m : C_n \to C_n$ denote the homomorphism $\varphi_m(x^t) = x^{mt}$. State a necessary and sufficient condition on m for φ_m to be an automorphism.

   Describe $\text{Aut}(C_{30})$ explicitly as a product of cyclic groups.

2. Let $\circ : G \times X \to X$ be a left action of a finite group G on a finite set X, and let $x \in X$. Explain what is meant by

   i) the orbit, $\langle x \rangle$, of $x \in X$;

   ii) the stability subgroup G_x.

   Prove that

   iii) if $y \in X$ then *either* $\langle x \rangle = \langle y \rangle$ *or* $\langle x \rangle \cap \langle y \rangle = \emptyset$, and

   iv) show there exists a bijection $\langle x \rangle \leftrightarrow G/G_x$.

   Explain what is meant by the Class Equation of such an action, and describe it explicitly in the case where $X = G = A_4$, the alternating group of order 12, and the action is *conjugation* $\circ : A_4 \times A_4 \to A_4$; $g \circ h = ghg^{-1}$.

3. Let p be a prime and P a group of order p^n acting on a finite set X with fixed point set X^P. Prove that $|X^P| \equiv |X| \pmod p$.

   Let G be a group of order kp^n where k is coprime to p, and let N_p be the number of subgroups of order p^n. Under the assumption that $N_p \neq 0$, show that

   $$N_p \equiv 1 \pmod p.$$

   Suppose that the prime p has the form $p = 2^a - 1$; assuming still that $N_p \neq 0$ deduce that if G is a group of order $2^a p$ then G has *either*

   i) a normal subgroup of order p *or* ii) a normal subgroup of order 2^a.

4. State Sylow's Theorem.

Let p, q be primes such that $q^n < p$ and let G be a group of order pq^n. Assuming Sylow's Theorem, prove that G is a semi-direct product

$$G \cong P \rtimes Q$$

where $|P| = p$ and $|Q| = q^n$.

Use this result to describe all groups of order 207, stating with justification the number of distinct isomorphism types obtained.

5. Let A be a commutative integral domain which contains a field $\mathbb{F}$ as a subring and is such that $\dim_{\mathbb{F}}(A)$ is finite. Show that A is a field.

Deduce that if $p(x)$ is an irreducible polynomial over a field $\mathbb{F}$ then $\mathbb{F}[x]/(p(x))$ is a field.

If $\mathbb{F}_3$ denotes the field with three elements, show that

i) $x^2 + 1$ and $x^2 + x + 2$ are both irreducible over $\mathbb{F}_3$, and that

ii) there is an isomorphism of fields $\mathbb{F}_3[x]/(x^2 + 1) \cong \mathbb{F}_3[x]/(x^2 + x + 2)$.

6. State and prove Eisenstein's Criterion.

Give the complete factorizations of the polynomials below into monic irreducible factors over $\mathbb{Q}$, justifying your answer in each case.

i) $x^{12} - 89x^8 - 1600$;

ii) $x^{15} + 1$.

END OF PAPER