UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:–

B.Sc. *M.Sci.*

Mathematics M222: Algebra 4: Groups and Rings

COURSE CODE	:	**MATHM222**
UNIT VALUE	:	**0.50**
DATE	:	**16–MAY–05**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

05–C0964–3–90

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Let G be a finite group. Explain what is meant by the order, $\text{ord}(g)$, of $g \in G$.

   If x is a generator of the cyclic group C_n of order n, show that, for $1 \leqslant m \leqslant n-1$,

$$\text{ord}(x^m) = \frac{n}{\text{HCF}(m,n)}.$$

   Let $\varphi_m : C_n \to C_n$ denote the homomorphism $\varphi_m(x^t) = x^{mt}$. Derive a necessary and sufficient condition on m for φ_m to be an automorphism.

   Define the *Euler totient function* $\Phi(n)$, and prove that if $p_1, \ldots, p_m$ are the distinct primes dividing the positive integer n then

$$\Phi(n) = n\left(1 - \frac{1}{p_1}\right)\cdots\left(1 - \frac{1}{p_m}\right).$$

   Hence or otherwise, find the order of $\text{Aut}(C_{2700})$.

2. Let K be a group ; explain how the set $\text{Aut}(K)$ of automorphisms of K forms a group, and describe explicitly

   (i) the group structure on $\text{Aut}(C_{13})$ and

   (ii) all group homomorphisms $h : C_3 \to \text{Aut}(C_{13})$.

   If $h : Q \to \text{Aut}(K)$ is a group homomorphism, explain what is meant by the *semi-direct product*

$$K \rtimes_h Q\,.$$

   If G is a finite group with subgroups K and Q, state and prove a criterion which allows one to assert that G is a semi-direct product of the above form.

   Decide with proof how many isomorphically distinct groups there are of the form

$$C_{13} \rtimes_h C_3\,.$$

   and describe each of them in terms of generators and relations.

3. Let $\circ : G \times X \to X$ be a left action of a finite group G on a finite set X. Explain what is meant by

   i) the orbit $\langle x \rangle$ of $x \in G$;

   ii) the stability subgroup G_x of $x \in X$;

   iii) the quotient set G/G_x.

   Show that, for any $x \in X$ there exists a bijection

$$\langle x \rangle \longleftrightarrow G/G_x$$

   Explain what is meant by the *class equation* of such an action (in each of its forms).

   Furthermore describe the class equation explicitly in the case where $X = G = D_{10}$, the dihedral group of order 10, and the action is *conjugation*

$$\circ \ : \ D_{10} \times D_{10} \to D_{10} \ ; \ g \circ h = ghg^{-1}.$$

4. Let p be a prime, and let G be a group of order p^n $(n \geqslant 1)$ acting on a finite set X. Define the *fixed point set* X^G, and prove that

$$|X| \equiv |X^G| \pmod p.$$

   For any integer $k \geqslant 1$, by means of a suitable action of G show that

$$\binom{kp^n}{p^n} \equiv k \pmod p.$$

   Let p be a prime, and let G be a group of order kp^n where $n \geqslant 1$ and k is coprime to p, and let N_p be the number of subgroups of G of order p^n. Assuming that $N_p \geqslant 1$, and stating clearly any auxiliary results that you employ, show that

$$N_p \equiv \ 1 \pmod p.$$

5. Let $\mathbf{F}$ be a field and let G be a finite subgroup of the multiplicative group $\mathbf{F}^*$. Prove that G is cyclic.

Show that $p(x) = x^2 + 3x + 3$ is irreducible over the field $\mathbf{F}_5$.

Let G denote the unit group

$$G = [\mathbf{F}_5[x]/(x^2 + 3x + 3)]^*.$$

By showing that $x^6 = 3$ in $\mathbf{F}_5[x]/(x^2 + 3x + 3)$, or otherwise, describe an explicit isomorphism $G \cong C_n$, for some n, and state the value of n.

6. Let $a(x)$ be a polynomial with integer ocefficients. Define the *content* $C(a)$ of $a(x)$.

If $b(x)$ is also an integral polynomial prove that

$$C(a) = C(b) = 1 \implies C(ab) = 1.$$

In each case below, decide whether or not the given polynomial is irreducible over $\mathbf{Q}$, jusifying your answer in each case. If the polynomial is not irreducible, give its complete factorization into $\mathbf{Q}$-irreducible factors.

(i) $x^{12} - 18x^6 - 175$;

(ii) $x^6 + x^3 + 1$.

(iii) $x^{10} + 1$.