UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M222: Algebra 4: Groups and Rings

COURSE CODE	:	**MATHM222**
UNIT VALUE	:	**0.50**
DATE	:	**13-MAY-04**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

04-C0958-3-90

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Let G be a finite group. Explain what is meant by the order, ord g , of $g \in G$.
   Define the *kernel* $\mathrm{Ker}(\varphi)$ and *image* $\mathrm{Im}(\varphi)$ of a group homomorphism $\varphi : G \to H$.
   State and prove a relationship which holds between $\mathrm{Ker}(\varphi)$ and $\mathrm{Im}(\varphi)$.
   Deduce that if $x \in G$ then ord $\varphi(x)$ divides both $|G|$ and $|H|$.
   Let $\varphi_m : C_n \to C_n$ denote the homomorphism $\varphi_m(x^t) = x^{mt}$. State a necessary and sufficient condition on m for φ_m to be an automorphism.
   Describe $\mathrm{Aut}(C_{15})$ explicitly as a product of cyclic groups.

2. Let K, Q be groups. Explain what is meant by a *semi-direct product*

$$K \rtimes_{\varphi} Q.$$

   If K , Q are subgroups of the finite group G, state and prove a criterion which allows one to recognize G as such a semidirect product.
   List all homomorphisms $\varphi : C_5 \to \mathrm{Aut}(C_{22})$.
   Hence list, with explanation, the isomorphically distinct groups of the form

$$C_{22} \rtimes_{\varphi} C_5 .$$

3. Let $\circ : G \times X \to X$ be a left action of a finite group G on a finite set X, and let $x \in X$. Explain what is meant by
   (i) the fixed point set X^G ; (ii) the orbit $\langle x \rangle$; (iii) the stability subgroup $\mathrm{Stab}_G(x)$.
   Explain, with proof, what is meant by the Class Equation of such an action in both its set-theoretic and numerical forms.
   Describe the numerical form explicitly when $X = G = A_4$, the alternating group on 4 letters, and the action is *conjugation* $\circ : A_4 \times A_4 \to A_4$; $g \circ h = ghg^{-1}$.
   Let p be a prime and P a group of order p^n acting on a finite set X with fixed point set X^P. Prove that $|X^P| \equiv |X| \pmod p$.

4. Let p be a prime, and let G be a group of order kp^n where $n \geqslant 1$ and k is coprime to p, and let N_p be the number of subgroups of G of order p^n. Assuming that $N_p \geqslant 1$, show that

$$N_p \equiv 1 \pmod{p}.$$

Let p, q be primes such that $q^n < p$ and let G be a group of order pq^n. Stating any further assumptions you are making, prove that G is a semi-direct product

$$G \cong P \rtimes Q$$

where $|P| = p$ and $|Q| = q^n$.

Hence describe all groups of order 725.

5. Let $p(x)$ be an irreducible polynomial of degree $n \geqslant 1$ over a field $\mathbb{F}$; show that $\mathbb{F}[x]/(p(x))$ is an integral domain.

State a relationship between $\dim_{\mathbb{F}} \mathbb{F}[x]/(p(x))$ and n, and explain why $\mathbb{F}[x]/(p(x))$ is a field.

Let $\mathbb{F}_3$ denote the field with three elements ;

i) show that $x^2 + x + 2$ is irreducible over $\mathbb{F}_3$;

ii) by finding a generator, show explicitly that the multiplicative group $(\mathbb{F}_3[x]/(x^2 + x + 2))^*$ is cyclic.

6. State and prove Eisenstein's Criterion.

Give the complete factorisations of the polynomials below into monic irreducible factors over $\mathbb{Q}$, justifying your answer in each case.

i) $x^{22} - 3x^{11} + 2$;

ii) $x^4 - 4x^3 + 6x^2 + x + 1$.

Write down the complete factorisations of $x^{15} - 1$ and $x^{30} - 1$ into monic irreducible factors over $\mathbb{Q}$. Hence or otherwise, give the corresponding factorisation of $x^{15} + 1$.