UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M222: Algebra 4: Groups and Rings

COURSE CODE	:	**MATHM222**
UNIT VALUE	:	**0.50**
DATE	:	**30-APR-03**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

03-C0951-3-60

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Let K be a group ; describe the group structure on the set $\mathrm{Aut}(K)$ of automorphisms of K.

   If $\varphi : Q \to \mathrm{Aut}(K)$ is a group homomorphism, explain what is meant by the *semi-direct product*

   $$K \rtimes_{\varphi} Q ,$$

   and show it is a group.

   Describe explicitly

   (i) the group structure on $\mathrm{Aut}(C_{28})$;

   (ii) all homomorphisms $\varphi : C_3 \to \mathrm{Aut}(C_{28})$.

   How many isomorphically distinct groups are there of the form

   $$C_{28} \rtimes_{\varphi} C_3 \quad ?$$

2. Let $\circ : G \times X \to X$ be a left action of a finite group G on a finite set X, and let $x \in X$. Explain what is meant by

   (i) the orbit, $\langle x \rangle$, of x ;

   (ii) the *stability group* $\mathrm{Stab}_G(x)$ of x .

   Explain, with proof, what is meant by the *class equation* of the action in both its set-theoretic and numerical forms.

   Let $G = A_4$, the alternating group on 4 letters, and let G act on itself by *conjugation*

   $$\circ \; : \; A_4 \times A_4 \to A_4 \; ; \; g \circ h = ghg^{-1}.$$

   Give an explicit description of

   (i) the orbits in this action ;

   (ii) the stability subgroup of a representative element in each orbit ;

   (iii) both forms of the class equation.

3. Let p be a prime, and let G be a group of order p^n ($n \geqslant 1$) acting on a finite set X. Define the *fixed point set* X^G, and prove that

$$|X| \equiv |X^G| \pmod{p}.$$

Deduce that the centre $Z(G)$ of G is nontrivial.

By means of a suitable action, for any integer $k \geqslant 1$ show that

$$\binom{kp^n}{p^n} \equiv k \pmod{p}.$$

4. Let P, Q be subgroups of a group G; explain what is meant by saying that P *normalizes* Q. Show that, when P normalizes Q, there is a group isomorphism

$$PQ/Q \cong P/(P \cap Q).$$

Let p be a prime, and let G be a group of order kp^n where $n \geqslant 1$ and k is coprime to p, and let N_p be the number of subgroups of G of order p^n. Assuming that $N_p \geqslant 1$, show that

$$N_p \equiv 1 \pmod{p}.$$

Deduce that if G is a group of order 153 then

(i) G has a normal subgroup of order 17 ;

(ii) G is abelian.

How many isomorphically distinct groups of order 153 are there ?

5. Let $\mathbf{F}$ be a field and let G be a finite subgroup of the multiplicative group $\mathbf{F}^*$. Prove that G is cyclic.

Show that $p(x) = x^2 + 3x + 3$ is irreducible over the field $\mathbf{F}_5$.

Let G denote the unit group

$$G = [\mathbf{F}_5[x]/(x^2 + 3x + 3)]^*.$$

Show that $G \cong C_n$, for some n, and state the value of n.

By showing that $x^6 = 3$ in $\mathbf{F}_5[x]/(x^2 + 3x + 3)$, or otherwise, find a generator for G.

6. State Eisenstein's criterion for irreducibility.

State and prove Gauss' Lemma.

In each case below, decide whether or not the given polynomial is irreducible over $\mathbf{Q}$;

(i) $x^4 + 7x^3 + 21x^2 + 22x + 10$;

(ii) $x^{16} - 16x^8 + 63$.

If the polynomial is not irreducible, give its complete factorization into $\mathbf{Q}$-irreducible factors.