UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Sc. *M.Sci.*

Mathematics M222: Algebra 4: Groups and Rings

COURSE CODE	:	**MATHM222**
UNIT VALUE	:	**0.50**
DATE	:	**01-MAY-02**
TIME	:	**14.30**
TIME ALLOWED	:	**2 hours**

02-C0946-3-70

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Let G be a finite group. Explain what is meant by the order, $\mathrm{ord}(g)$, of $g \in G$.

   If x is a generator of the cyclic group C_n of order n, show that, for $1 \leqslant m \leqslant n-1$,

$$\mathrm{ord}(x^m) = \frac{n}{\mathrm{HCF}(m,n)}.$$

   Let $\varphi_m : C_n \to C_n$ denote the homomorphism $\varphi_m(x^t) = x^{mt}$. Derive a necessary and sufficient condition on m for φ_m to be an automorphism.

   Describe $\mathrm{Aut}(C_{28})$ explicitly as a product of cyclic groups.

2. Let $\circ : G \times X \to X$ be a left action of a finite group G on a finite set X, and let $x \in X$. Explain what is meant by

   i) the orbit, $\langle x \rangle$, of $x \in X$;

   ii) the stability subgroup $\mathrm{Stab}_G(x)$.

   Prove that

   iii) $|\langle x \rangle| = |G|/|\mathrm{Stab}_G(x)|$, and also

   iv) if $y \in X$ then *either* $\langle x \rangle = \langle y \rangle$ *or* $\langle x \rangle \cap \langle y \rangle = \emptyset$.

   Explain what is meant by the Class Equation of such an action, and describe it explicitly in the case where $X = G = D_6^*$, the binary dihedral group of order 12, and the action is *conjugation* $\circ : D_6^* \times D_6^* \to D_6^*$; $g \circ h = ghg^{-1}$.

3. Let p be a prime and P a group of order p^n acting on a finite set X with fixed point set X^P. Prove that $|X^P| \equiv |X| \pmod p$.

   Let G be a group of order kp^n where k is coprime to p, and let N_p be the number of subgroups of order p^n. Under the assumption that $N_p \neq 0$, show that

$$N_p \equiv 1 \pmod p.$$

   Maintaining this assumption, deduce that if G is a group of order 12 then G has *either* a normal subgroup of order 3 *or* a normal subgroup of order 4.

4. State Sylow's Theorem.

   Let p, q be primes such that $q^n < p$ and let G be a group of order pq^n. Assuming Sylow's Theorem, prove that G is a semi-direct product

$$G \cong P \rtimes Q$$

   where $|P| = p$ and $|Q| = q^n$.

   Use this result to describe all groups of order 76.

   How many distinct isomorphism types of such groups are there ?

   Justify your statement.

5. Let A be a commutative integral domain which contains a field $\mathbb{F}$ as a subring and is such that $\dim_{\mathbb{F}}(A)$ is finite. Show that A is a field.

   Deduce that if $p(x)$ is an irreducible polynomial over a field $\mathbb{F}$ then $\mathbb{F}[x]/(p(x))$ is a field.

   If $\mathbb{F}_3$ denotes the field with three elements, show that

   i) $x^2 + 1$ and $x^2 + x + 2$ are both irreducible over $\mathbb{F}_3$, and that

   ii) there is an isomorphism of fields $\mathbb{F}_3[x]/(x^2 + 1) \cong \mathbb{F}_3[x]/(x^2 + x + 2)$.

6. State and prove Eisenstein's Criterion.

   Give the complete factorisations of the polynomials below into monic irreducible factors over $\mathbb{Z}$, justifying your answer in each case.

   i) $x^{30} - x^{15} - 6$;

   ii) $x^4 + 4x^3 + 6x^2 + 13x + 13$;

   iii) $x^{15} - 1$.