

UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:–

B.Sc. *M.Sci.*

Mathematics M12B: Algebra 2

COURSE CODE : **MATHM12B**

UNIT VALUE : **0.50**

DATE : **16–MAY–06**

TIME : **14.30**

TIME ALLOWED : **2 Hours**

All questions may be attempted but only marks obtained on the best four solutions will count.

The use of an electronic calculator is **not** permitted in this examination.

1. (a) Give the definition of a *group*, defining the terms you use. Prove that in any group the identity element is unique, and each element has a unique inverse.
(b) Determine whether or not the following sets G under the given operation \star are groups, justifying your answer:
 - (i) $G = \mathbb{R} - \{-2\}$, $a \star b = ab + 2a + 2b + 2$,
 - (ii) $G = \mathbb{R}$, $a \star b = a - b$,
 - (iii) $G = \{x \in \mathbb{R} : x \geq 0\}$, $a \star b = +\sqrt{a^2 + b^2}$.
2. (a) State (do not prove) Lagrange's Theorem. Hence prove that in a finite group G the order of any element divides the order of the group.
(b) Deduce that for any prime p if $a \not\equiv 0 \pmod{p}$ then $a^{p-1} \equiv 1 \pmod{p}$.
(c) Find $2^{3599} \pmod{37}$.
(d) Find an number x such that $x^7 \equiv 2 \pmod{37}$
3. (a) Stating clearly any results you use, prove that for any two $n \times n$ matrices A and B , $\det(AB) = \det(A) \det(B)$.
(b) Find $\det \begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix}$, expressing your answer as a product of linear factors.

MATHM12B

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4. (a) Let A be an $n \times n$ matrix over \mathbb{R} . Give the definition of:
- (i) an *eigenvalue* λ of A ;
 - (ii) an *eigenvector* corresponding to λ ;
 - (iii) the *eigenspace* E_λ of A ;
 - (iv) A is *diagonalizable* (over \mathbb{R}). State (do not prove) the basic criterion for a matrix to be diagonalisable.
- (b) Let λ_i ($i = 1, \dots, r$) be the distinct eigenvalues of A ; prove that the sum $\sum_{i=1}^r E_{\lambda_i}$ is direct. Deduce that if $\sum_{i=1}^r \dim(E_{\lambda_i}) = n$ then A is diagonalisable.
- (c) Prove that the following matrix is diagonalisable:

$$\begin{pmatrix} 2 & 0 & 2 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ -2 & 0 & -2 & -2 \end{pmatrix}$$

5. Let $A = \begin{pmatrix} 3/2 & -1 \\ 1/2 & 0 \end{pmatrix}$.

- (i) Find an invertible matrix P such that $P^{-1}AP$ is diagonal.
- (ii) Find A^n for $n \in \mathbb{N}$
- (iii) Solve the system of difference equations

$$\begin{aligned} x_{n+1} &= \frac{3}{2}x_n - y_n \\ y_{n+1} &= \frac{1}{2}x_n \end{aligned}$$

given that $x_0 = 0, y_0 = 1$.

Find the limit, as $n \rightarrow \infty$ of x_n .

6. (a) Let A be a real symmetric matrix and let \mathbf{u} , \mathbf{v} be eigenvectors associated to the (real) eigenvalues λ and μ respectively, where $\lambda \neq \mu$. Prove that \mathbf{u} and \mathbf{v} are orthogonal vectors.

(b) Let $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. Find an orthogonal matrix P such that $P^{-1}AP$ is diagonal.