UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. B.Sc.(Econ)M.Sci.

Mathematics M12B: Algebra 2

COURSE CODE : **MATHM12B**

UNIT VALUE : **0.50**

DATE : **09-MAY-05**

TIME : **14.30**

TIME ALLOWED : **2 Hours**

05-C0956-3-200

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*

The use of an electronic calculator is **not** *permitted in this examination.*

1. Let H be a subset of a group G. Give necessary and sufficient conditions for H to be a subgroup of G. In each of the following cases, determine if H is a subgroup of G or not, justifying your answer:

   (i) $G = \mathbb{R}$ (under addition), $H = \{x \in G : x \geq 0\}$;

   (ii) $G = GL_2(\mathbb{R})$, $H = \{A \in G : A^{-1} = A^T\}$;

   (iii) $G = S(\mathbb{R})$, $H = \{f \in G : f(1) = 1\}$;

   (iv) G is any abelian group, $H = \{g \in G : g^2 = e\}$;

   (v) $G = S_7$, $H = \{g \in G : g^2 = e\}$.

   [$GL_2(\mathbb{R})$ denotes the group of real 2×2 invertible matrices under matrix multiplication: $S(\mathbb{R})$ is the group of bijections from $\mathbb{R}$ to $\mathbb{R}$ under composition; S_7 is the group of permutations of 1,2,3,4,5,6,7]

2. (a) State, without proof, Lagrange's Theorem. Prove that in a finite group G the order of any element divides the order of the group.

   (b) Deduce that $\overline{a}^{p-1} = \overline{1}$ in $\mathbf{Z}_p^*$ for all $\overline{a} \in \mathbf{Z}_p^*$ (where p is a prime and $\mathbf{Z}_p^*$ denotes the group of non-zero integers mod p under multiplication).

   (c) Find (i) $\overline{2}^{1803}$, (ii) $\overline{2}^{358}$ in $\mathbf{Z}_{19}^*$.

   (d) Show that every element in $\mathbf{Z}_{19}^*$ has a 5th root.

3. (a) Let A be an $n \times n$ matrix. Give the definition of $\det(A)$. State, without proof, the effect on the determinant of each type of elementary row operation. Give a formula for the determinant of an upper triangular matrix and prove it.

   (b) Evaluate $\det \begin{pmatrix} -1 & 2 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ -1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 3 & -1 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$.

   (c) Find $\det \begin{pmatrix} a & b & c & d \\ b & c & d & a \\ c & d & a & b \\ d & a & b & c \end{pmatrix}$, expressing your answer as a product of linear and/or quadratic factors.

4. (a) Let A be an $n \times n$ matrix over $\mathbb{R}$. Give the definition of:

   (i) an *eigenvalue* λ of A;

   (ii) an *eigenvector* $\mathbf{v}$ of A;

   (iii) the *characteristic polynomial* $c_A(t)$ of A;

   (iv) A is *diagonalizable* (over $\mathbb{R}$).

   (b) Prove that if A has n distinct eigenvalues, then A is diagonalisable.

   (c) Let D be an $n \times n$ diagonal matrix with distinct entries on the diagonal, and X an $n \times n$ matrix such that $XD = DX$. Prove that X is diagonal.

   Let A and B be two $n \times n$ matrices, each of which has n distinct eigenvalues and such that $AB = BA$. Prove that they are simultaneously diagonalisable, i.e. there exists an invertible P such that $P^{-1}AP$ and $P^{-1}BP$ are both diagonal.

5. Let $A = \begin{pmatrix} 7 & -10 \\ 3 & -4 \end{pmatrix}$.

   (i) Find an invertible matrix P such that $P^{-1}AP$ is diagonal.

   (ii) Find A^n (for positive integers n).

   (iii) Solve the system of equations

$$\begin{aligned} \frac{dx_1}{dt} &= 7x_1 - 10x_2 \\ \frac{dx_2}{dt} &= 3x_1 - 4x_2 \end{aligned}$$

   given that $x_1(0) = 0$, $x_2(0) = 1$.

   (iv) Suppose a sequence of vectors $\mathbf{v}_i$ is given by $\mathbf{v}_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\mathbf{v}_{n+1} = A^{-1}\mathbf{v}_n$. Find the limit, as $n \longrightarrow \infty$, of $\mathbf{v}_n$.

6. (a) Let A be a real symmetric matrix and let $\mathbf{u}$, $\mathbf{v}$ be eigenvectors associated to the (real) eigenvalues λ and μ respectively, where $\lambda \neq \mu$. Prove that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors.

   (b) Let $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{pmatrix}$. Find an orthogonal matrix P such that $P^{-1}AP$ is diagonal.

   (c) Prove that if A is a real matrix which is orthogonally diagonalisable then A is symmetric.