UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M12B: Algebra 2

COURSE CODE	:	**MATHM12B**
UNIT VALUE	:	**0.50**
DATE	:	**07-MAY-04**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

04-C0950-3-190

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*

The use of an electronic calculator is **not** *permitted in this examination.*

1. Let G be a group and H a subset of G. State the conditions usually used to check that H is a subgroup of G. In each of the following cases, determine if H is a subgroup of G or not, justifying your answer:

   (i) $G = GL_2(\mathbf{R})$, $H = \{A \in G : A^{-1} = A\}$.

   (ii) $G = GL_2(\mathbf{R})$, $H = \{A \in G : A^{-1} = A^T\}$.

   (iii) $G = GL_2(\mathbf{R})$, $H = \{A \in G : A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, a \geq 0\}$.

   (iv) $G = S_3$, H is the set of elements of G of order 1 or 2.

   (v) $G = S_3$, H is the set of elements of G of order 1 or 3.

   [$GL_2(\mathbf{R})$ is the group of real 2×2 invertible matrices under multiplication: S_3 is the group of permutations of $\{1, 2, 3\}$.]

2. (a) Prove Lagrange's Theorem.

   (b) Deduce that a group of prime order is cyclic.

   (c) Let G be the symmetry group of a regular pentagon. Find all subgroups of G, justifying your answer.

   [You may assume, if you wish, that elements of G have a normal form $x^i y^j$ ($0 \leq i \leq 4$, $0 \leq j \leq 1$) and that G has presentation $< x, y : x^5 = y^2 = e, yx = xy^4 >$]

3. (a) Let A be an $n \times n$ matrix. Give the definition of $\det(A)$. Let B be obtained from A by performing the elementary row operation e.

   (i) Prove that if $e = p(r, s)$ (exchange rows r and s), then $\det(B) = -\det(A)$.

   (ii) State (do not prove) what happens if e is each of the other two types of elementary row operation. Also state (do not prove) what is the determinant of an upper triangular matrix.

   (b) Evaluate the determinant of each of the following matrices, in case (ii) giving the answer in factorized form:

   (i) $\det \begin{pmatrix} 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 2 & -2 & 3 \\ 2 & 0 & 0 & 1 & 1 \end{pmatrix}$ (ii) $\det \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{pmatrix}$

4. (a) Let A be an $n \times n$ matrix over $\mathbf{R}$. Give the definition of:

(i) an *eigenvalue* λ of A;

(ii) an *eigenvector* $\mathbf{v}$ of A;

(iii) the *characteristic polynomial* $c_A(t)$ of A;

(iv) A is *diagonalizable* (over $\mathbf{R}$).

Prove that A is diagonalizable if and only if $\mathbf{R}^n$ has a basis of eigenvectors of A.

(b) Prove that if A has n distinct eigenvalues, then A is diagonalisable.

(c) In each case below give an example of a 2×2 matrix with only one distinct eigenvalue such that

(i) A is diagonalizable;

(ii) A is not diagonalizable.

5. Let $A = \begin{pmatrix} -5 & 4 \\ -6 & 5 \end{pmatrix}$.

(i) Find an invertible matrix P such that $P^{-1}AP$ is diagonal.

(ii) Find a matrix B such that $B^3 = A$

(iii) Solve the system of equations

$$\begin{aligned} \frac{dx_1}{dt} &= -5x_1 + 4x_2 \\ \frac{dx_2}{dt} &= -6x_1 + 5x_2 \end{aligned}$$

given that $x_1(0) = 3$, $x_2(0) = 4$.

(iv) The exponential e^X for a square matrix X is defined by

$$e^X = \sum_{i=0}^{\infty} \frac{1}{i!} A^i$$

Find e^A for the above matrix A.

6. (a) Prove that if A is a real symmetric matrix then all the eigenvalues of A are real, and A is orthogonally diagonalisable.

(b) Let $A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$. Find an orthogonal matrix P such that $P^{-1}AP$ is diagonal.