UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M12B: Algebra 2

COURSE CODE	:	**MATHM12B**
UNIT VALUE	:	**0.50**
DATE	:	**08-MAY-03**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

03-C0943-3-180

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

Throughout $\mathbb{F}$ denotes a field and a basis is always assumed to contain a finite number of elements.

1. (a) Let $A = (a_{i,j}) \in {}^n\mathbb{F}^n$.
   Define
   (i) the (i,j) minor $M_{i,j}$ of A,
   (ii) the (i,j) co-factor $A_{i,j}$ of A.

   Give the expansions of $|A|$ by its ith row and by its jth column.
   Prove that $|A^T| = |A|$, where A^T is the transpose of A.

   (b) Define

$$A_n = \begin{bmatrix} -2 & 1 & \cdots & 0 & 0 \\ 1 & -2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -2 & 1 \\ 0 & 0 & \cdots & 1 & -2 \end{bmatrix}$$

   where $A_n \in {}^n\mathbb{R}^n$ has -2's down the main diagonal, 1's down the superdiagonals and zeros elsewhere.
   Let $u_n = |A_n|$, where $u_1 = -2$ and $u_2 = 3$. Prove that $u_n = -2u_{n-1} - u_{n-2}$ for $n \geqslant 3$ and deduce that $u_n = (-1)^n(n+1)$.

2. (a) Let $A \in {}^n\mathbb{F}^n$. Define the terms:

   (i) eigenvalue of A,

   (ii) eigenvector of A,

   (iii) A is diagonalizable.

   Show that A is diagonalizable if and only if there is an invertible matrix $P \in GL(n, \mathbb{F})$, whose columns are eigenvectors of A.

   (b) Let

$$A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix} \in {}^2\mathbb{R}^2.$$

   (i) Find $P \in GL(2, \mathbb{R})$ such that $P^{-1}AP$ is a diagonal matrix.

   (ii) Find A^n for every $n \in \mathbb{N}$.

3. Let ${}_{\mathbb{F}}V$ be a vector space and $U \subseteq V$. Define what it means to say that U is a **subspace** of V.

   (a) Let A and B be subspaces of V.

   (i) Define $A + B$ and show that $A + B$ is a subspace of V.

   (ii) Define what it means to say that V is **the direct sum of** A **and** B, $V = A \oplus B$. Show that $V = A \oplus B$ if and only if every $\underline{v} \in V$ can be expressed in the form $\underline{v} = \underline{a} + \underline{b}$ for unique $\underline{a} \in A$, $\underline{b} \in B$.

   (b) Let $A = \{(s, 2s) : s \in \mathbb{R}\}$ and $B = \{(t, t) : t \in \mathbb{R}\}$. Show that A and B are subspaces of $\mathbb{R}^2$ and that $\mathbb{R}^2 = A \oplus B$. Express $(-1, 1) \in \mathbb{R}^2$ uniquely in the form $(-1, 1) = \underline{a} + \underline{b}$, where $\underline{a} \in A, \underline{b} \in B$. **Justify your answers.**

[In your answers to (a) and (b) you may use any standard subspace tests which you require.]

4. (a) Let $\{\underline{v}_1, \ldots, \underline{v}_r\} \subseteq V$ where ${}_{\mathbb{F}}V$ is a vector space. Define the following:

   (i) $\{\underline{v}_1, \ldots, \underline{v}_r\}$ is linearly dependent,

   (ii) $\{\underline{v}_1, \ldots, \underline{v}_r\}$ is linearly independent,

   (iii) the linear span $L(\underline{v}_1, \ldots, \underline{v}_r)$ of $\{\underline{v}_1, \ldots, \underline{v}_r\}$,

   (iv) $\{\underline{v}_1, \ldots, \underline{v}_r\}$ is a spanning set for V,

   (v) $\{\underline{v}_1, \ldots, \underline{v}_r\}$ is a basis for V.

   Show that $\{\underline{v}_1, \ldots, \underline{v}_r\}$ is linearly dependent if and only if $\underline{v}_i \in L(\underline{v}_1, \ldots, \underline{v}_{i-1})$ for some $1 \leqslant i \leqslant r$.

   (b) Let $\underline{a}_i \in {}^5\mathbb{R}$, $1 \leqslant i \leqslant 5$, where

$$\underline{a}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \quad \underline{a}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \quad \underline{a}_3 = \begin{bmatrix} 3 \\ -1 \\ 0 \\ 4 \\ 3 \end{bmatrix}, \quad \underline{a}_4 = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \quad \underline{a}_5 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}.$$

   Show that $\{\underline{a}_1, \underline{a}_2, \underline{a}_3, \underline{a}_4, \underline{a}_5\}$ is linearly dependent and find $1 \leqslant i \leqslant 5$ such that $\underline{a}_i \in L(\underline{a}_1, \ldots, \underline{a}_{i-1})$. Find a subset of $\{\underline{a}_1, \underline{a}_2, \underline{a}_3, \underline{a}_4, \underline{a}_5\}$ which is a basis for $L(\underline{a}_1, \underline{a}_2, \underline{a}_3, \underline{a}_4, \underline{a}_5)$. **Justify your answers.**

5. State, **without proof**, the Steinitz Exchange theorem.

   (a) Show that any two bases of a vector space always have the same number of elements.

   State what it means to say that a vector space V is finite-dimensional and define the dimension of V.

   (b) Let ${}_{\mathbb{F}}V$ be a finite-dimensional vector space with $\dim(V) = n$. If $\{\underline{v}_1, \ldots, \underline{v}_n\} \subseteq V$ spans V, show that $\{\underline{v}_1, \ldots, \underline{v}_n\}$ is a basis for V.

   [**You may assume, without proof, any standard results about spanning sets.**]

   (c) Consider $\mathbb{R}_3[x] = \{a_0 + a_1x + a_2x^2 + a_3x^3 : a_i \in \mathbb{R}, 0 \leqslant i \leqslant 3\}$ as a vector space over $\mathbb{R}$ in the usual way. Determine which of the following sets of elements of $\mathbb{R}_3[x]$, if any, are bases for $\mathbb{R}_3[x]$.

   (i) $A = \{(1 + x), (x + x^2), (x^2 + x^3)\}$

   (ii) $B = \{(1 - x^3), (x - 1), (x^2 - x), (x^3 - x^2)\}$,

   (iii) $C = \{1, 1 + x, 1 + x + x^2, 1 + x + x^2 + x^3\}$.

6. Let ${}_{\mathbb{F}}V$, ${}_{\mathbb{F}}W$ be vector spaces. Define what it means to say that $\alpha : V \to W$ is a linear map. Show that this is the case if and only if $(\forall \lambda, \mu \in \mathbb{F})(\forall \underline{a}, \underline{b} \in V)\, \alpha(\lambda \underline{a} + \mu \underline{b}) = \lambda \alpha(\underline{a}) + \mu \alpha(\underline{b})$.

Let $\alpha : V \to W$ be a linear map. Define (i) $\mathrm{Im}(\alpha)$, (ii) $\mathrm{Ker}(\alpha)$ and show that $\mathrm{Im}(\alpha) \leqslant W$ and $\mathrm{Ker}(\alpha) \leqslant V$.

Suppose further that V and W are finite-dimensional vector spaces. Define (i) $r(\alpha)$, the rank of α, (ii) $n(\alpha)$, the nullity of α. State, **without proof**, a relation between $r(\alpha)$ and $n(\alpha)$.

For any $A, B \in {}^{2}\mathbb{F}^{2}$ define $\alpha : {}^{2}\mathbb{F}^{2} \to {}^{2}\mathbb{F}^{2}$ by $X \mapsto AX + XB$. Show that α is a linear map. For $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, find (i) $\mathrm{Ker}(\alpha)$, (ii) $\mathrm{Im}(\alpha)$, (iii) $n(\alpha)$, (iv) $r(\alpha)$. Verify the relation between $r(\alpha)$ and $n(\alpha)$ which you stated above. **Justify your answers.**