UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Sc. *M.Sci.*

Mathematics M12B: Algebra 2

COURSE CODE	:	**MATHM12B**
UNIT VALUE	:	**0.50**
DATE	:	**13-MAY-02**
TIME	:	**14.30**
TIME ALLOWED	:	**2 hours**

02-C0938-3-170

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

Throughout $\mathbb{F}$ denotes a field.

1. (a) Let $A = (a_{i,j}) \in {}^n\mathbb{F}^n$. Define
   (i) the (i,j) minor, $M_{i,j}$, of A,
   (ii) the (i,j) co-factor, $A_{i,j}$, of A,
   (iii) the adjugate, $Adj(A)$, of A.
   Explain how to expand $|A|$ by its ith row and jth column.
   Show that $A\ Adj(A) = |A|I_n = Adj(A)A$.

   (b) Let $A' \in {}^n\mathbb{F}^n$ be obtained from $A \in {}^n\mathbb{F}^n$ by an elementary row operation. For each such operation state, *without proof*, how to calculate $|A'|$ in terms of $|A|$.
   Let $u_n = |A_n|$ where

$$A_n = \begin{bmatrix} 2 & 1 & 1 & . & . & . & 1 & 1 \\ 1 & 2 & 1 & . & . & . & 1 & 1 \\ 1 & 1 & 2 & . & . & . & 1 & 1 \\ . & . & . & & & & . & . \\ . & . & . & & & & . & . \\ . & . & . & & & & . & . \\ 1 & 1 & 1 & . & . & . & 2 & 1 \\ 1 & 1 & 1 & . & . & . & 1 & 2 \end{bmatrix}$$

   where the (i,j) entry of A is

$$\begin{cases} 1 & \text{if } i \neq j \\ 2 & \text{if } i = j \end{cases},$$

   $1 \leqslant i,\, j \leqslant n$. Prove that $u_n = n + 1$.

2. (a) Let $A \in {}^n\mathbb{F}^n$. Define the following terms:

   (i) eigenvalue of A,

   (ii) eigenvector of A,

   (iii) characteristic polynomial, $c_A(x)$, of A.

   If $B = A^T$ is the transpose of A, show that $c_B(x) = c_A(x)$.

   (b) Let $A = \begin{bmatrix} 6 & 5 \\ 3 & 4 \end{bmatrix} \in {}^2\mathbb{R}^2$.

   (i) Find $P \in GL(2, \mathbb{R})$ such that $P^{-1}AP$ is a diagonal matrix.

   (ii) Find A^n for every $n \in \mathbb{N}$.

   (iii) Find four distinct matrices $B \in {}^2\mathbb{R}^2$ such that $B^2 = A$.

3. Let ${}_\mathbb{F}V$ be a vector space.

   (a) Let $U \subseteq V$. Define what it means to say that U is a subspace of V. Show that this is the case if and only if U is non-empty and for every λ, $\mu \in \mathbb{F}$ and for every $\underline{a}$, $\underline{b} \in U$, $\lambda\underline{a} + \mu\underline{b} \in U$.

   (b) Let U and W be subspaces of V. Define $U + W$. Show that $U \cap W$ and $U + W$ are subspaces of V.

   (c) Let A, B, C be subspaces of V. Show that $(A \cap B) + (A \cap C) \subseteq A \cap (B + C)$. Give an example of subspaces A, B, C of $\mathbb{R}^2$ such that $(A \cap B) + (A \cap C) \neq A \cap (B + C)$. Justify your answer.

4. Let $\{\underline{v}_1, \ldots, \underline{v}_r\} \subseteq V$ where ${}_{\mathbb{F}}V$ is a vector space. Define the following:

   (i) $\{\underline{v}_1, \ldots, \underline{v}_r\}$ is linearly dependent,

   (ii) $\{\underline{v}_1, \ldots, \underline{v}_r\}$ is linearly independent,

   (iii) the linear span, $L(\underline{v}_1, \ldots, \underline{v}_r)$, of $\{\underline{v}_1, \ldots, \underline{v}_r\}$,

   (iv) $\{\underline{v}_1, \ldots, \underline{v}_r\}$ is a spanning set for V,

   (v) $\{\underline{v}_1, \ldots, \underline{v}_r\}$ is a basis for V.

   Show that some subset of $\{\underline{v}_1, \ldots, \underline{v}_r\}$ is a basis for $L(\underline{v}_1, \ldots, \underline{v}_r)$.

   [*Any standard results about linear dependence and spanning sets which you use in your proof should be carefully stated.*]

   Find a subset of the columns of the matrix

$$A = \begin{bmatrix} 1 & -2 & -1 & 1 & -1 \\ -1 & 2 & 1 & 1 & 3 \\ 2 & -3 & 0 & 1 & 0 \\ 1 & -1 & 1 & 1 & 2 \end{bmatrix} \in {}^4\mathbb{R}^5$$

   which is a basis for the column-space of A. Express each of the remaining columns of A as a linear combination of the members of this basis. Justify your answers.

5. State, *without proof*, the Steinitz exchange theorem. Show that any two bases for a vector space ${}_{\mathbb{F}}V$ have the same number of elements. Define the dimension of V.

   (a) Let $\{\underline{u}_1, \ldots, \underline{u}_m\} \subseteq V$ be linearly independent where ${}_{\mathbb{F}}V$ is a vector space with $\dim(V) = n$.
   Show that (i) $m \leqslant n$, (ii) if $m = n$, then $\{\underline{u}_1, \ldots, \underline{u}_m\}$ is a basis for V.

   (b) Let $\{\underline{e}_1, \ldots, \underline{e}_n\}$ be a basis for the vector space ${}_{\mathbb{R}}V$.
   If n is odd, show that $\{\underline{e}_1 + \underline{e}_2, \underline{e}_2 + \underline{e}_3, \ldots, \underline{e}_{n-1} + \underline{e}_n, \underline{e}_n + \underline{e}_1\}$ is linearly independent and a basis for V.

6. (a) Let $\alpha : V \to W$ where ${}_{\mathbb{F}}V$, ${}_{\mathbb{F}}W$ are vector spaces. Define

   (i) $\mathrm{Im}(\alpha)$,

   (ii) $\mathrm{Ker}(\alpha)$.

   Suppose in addition that ${}_{\mathbb{F}}V$, ${}_{\mathbb{F}}W$ are finite-dimensional. Define

   (i) the rank of α, $r(\alpha)$,

   (ii) the nullity of α, $n(\alpha)$.

   Prove that $r(\alpha) + n(\alpha) = \dim(V)$.

   (b) Let $\alpha : V \to V$ be a linear map where ${}_{\mathbb{F}}V$ is a finite-dimensional vector space. Suppose that $\alpha^2 = 0$. Show that $\mathrm{Im}(\alpha) \subseteq \mathrm{Ker}(\alpha)$ and hence or otherwise show that $r(\alpha) \leqslant \frac{1}{2}\dim(V)$.

   Give a specific example of α and V where $\alpha^2 = 0$ and $r(\alpha) = \frac{1}{2}\dim(V)$.

   [*Throughout you may assume any standard results about vector spaces and linear maps.*]