You should attempt all questions and show all working: stating the answers without showing how they were obtained will not attract credit.

1. (a) Find the second derivatives of $f(x) = \int_{-\infty}^{x^2/2} e^{x-t^2/2} dt$ and $g(x) = \int_{-\infty}^{x^2/2} e^{-(x^2+1)t^2} dt$.

(b) Derive the solution of the ordinary differential equation

$$\frac{d^2y}{dx^2} = f(x), \quad x > 0, \qquad y(0) = 0, \quad \frac{dy}{dx}(0) = 0,$$

in the form

$$y(x) = \int_0^x (x-t)f(t)\,dt.$$

(c) Find the three second partial derivatives of $f(x,y) = e^{-\frac{(x-1)^2-(y+1)^2}{2}}$.

2. Consider the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

By assuming a solution of the form $u(x,t) = X(x)T(t)$, deduce that

$$u_\alpha(x,t) = (A_\alpha \cos(\alpha x) + B_\alpha \sin(\alpha x))\, e^{-\alpha^2 t},$$

where A_α and B_α are constants, is a solution for any constant α. Show that if we now impose the boundary conditions $u(0,t) = 0$, $u(\pi,t) = 0$, then this reduces the possible solutions to those of the form

$$u_n(x,t) = B_n \sin(n\,x) e^{-n^2 t}$$

for $n = 0, 1, 2, \ldots$ and where B_n is a constant. Hence or otherwise find the solution of the problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = 0, \quad u(\pi,t) = 0$$
$$u(x,0) = \sin^2(x), \quad 0 < x < \pi.$$

3. By considering $S_n = 1 + x + x^2 + \cdots + x^n$, or otherwise, show that for $|x| < 1$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots .$$

Use this result to find the Taylor series of the functions below and indicate the values of x for which the corresponding series converges:

$$\text{(a)}\ \frac{1}{2-x}, \quad \text{(b)}\ \frac{x}{1+x-2x^2}, \quad \text{(c)}\ \log\left(\frac{1+x}{1-x}\right).$$

4. For the following matrix, find all eigenvalues and all (normalised) eigenvectors:

$$\begin{pmatrix} 6 & -2 & 2 \\ -2 & 5 & 0 \\ 2 & 0 & 7 \end{pmatrix}.$$

5. Solve the following ordinary differential equation initial value problems for $y(x)$:

(a) $y' + xy = 0, \quad y(0) = 1,$

(b) $x^2 y'' - 4xy' + 6y = 6, \quad y(1) = 0, y'(1) = 1,$

(c) $y'' + y' - 6y = 1, \quad y(0) = 0, \quad y'(0) = 2$

6. For the following functions, find the critical points, determine their nature (maxima, minima, inflection, etc.) and sketch the graph or surface:

(a) $f(x) = \dfrac{x}{1+x^2}$ for $x \neq 0$,

(b) $g(x) = |x| e^{-|x-1|}$,

(c) $F(x, y) = (x^2 - 4)^2 + y^2$,

(d) $G(x, y) = \sin x \ \sin y \ \sin(x + y) \quad (0 \leq x, y \leq \pi)$.

7. Consider the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

Show that if $u(x,t) = t^\alpha \phi(\xi)$ where $\xi = x/\sqrt{t}$ and α is a constant, then $\phi(\xi)$ satisfies the ordinary differential equation

$$\alpha\phi - \tfrac{1}{2}\xi\phi' = \phi'',$$

(where $' \equiv d/d\xi$). Show that

$$\int_{-\infty}^{\infty} u(x,t)\,dx = \int_{-\infty}^{\infty} t^\alpha \phi(\xi)\,dx$$

is independent of t only if $\alpha = -\frac{1}{2}$. Further, show that if $\alpha = -\frac{1}{2}$ then

$$C - \tfrac{1}{2}\xi\phi = \phi'$$

where C is an arbitrary constant. From this last ordinary differential equation, and assuming $C = 0$, deduce that

$$u(x,t) = \frac{A}{\sqrt{t}} e^{-x^2/4t}$$

is a solution of the heat equation (here A is an arbitrary constant).

Show that as t tends to zero from above,

$$\lim_{t\to 0+} \frac{1}{\sqrt{t}} e^{-x^2/4t} = 0 \text{ for } x \neq 0$$

and that for all $t > 0$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{t}} e^{-x^2/4t} dx = B$$

where B is a (finite) constant. Given that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$, find B.

What physical and/or probabilistic interpretation might one give to this solution $u(x,t)$?

8. Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables, each having a distribution function $F_X(x)$. Let $M = \min\{X_1, X_2, \ldots, X_n\}$. Find the distribution function of M. Now suppose F_X is the uniform distribution over $(0,1)$. What is the probability density function of M?

9. An infinite grid in the (x,y)-plane consists of lines $x = a*n$ and $y = b*n$, $n = 0, \pm 1, \pm 2, \ldots$ A needle of length $l < \min(a,b)$ is cast randomly onto the plane, without any preferred position or direction. What is the probability that it intersects a line?

10. Let X and Y be random variables with joint probability density

$$f_{X,Y}(x,y) = \begin{cases} ce^{-x-y}, & 0 < x < y < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Determine the value of the constant c. Determine whether X and Y are independent.

11. A gambler plays a game in which there is a probability p of winning one unit and probability $q = 1 - p$ of losing one unit. Successive plays of the game are independent. What is the probability that, starting with $x > 0$ units, the gambler's fortune will reach N? If $p = q$, find the expected time for a gambler who starts with $x > 0$ units to lose them all.

12. Let X and Y have a bivariate normal distribution.

(a) Show that X and Y are normally distributed, and find their mean and variance.

(b) Show that $X + Y$ is normally distributed, and find its mean and variance.

(c) Show that the conditional density of X given that $Y = y$ is normal, find its mean and variance.

13. The random variable Y has the standard normal density with mean 0 and variance 1, $Y \sim \mathcal{N}(0,1)$. Find the distribution and density functions of $V = Y^2$.

The moment generating function $M_X(t)$ of a random variable X is defined by $M_X(t) = E\left[e^{tX}\right]$. If $X \sim \mathcal{N}(0,1)$, show that $M_X(t) = e^{t^2/2}$.

Let X and Y be independent standard normal random variables, and $Z = XY$. Find $M_Z(t)$, either by direct calculation or via the tower law in the form $E\left[e^{tZ}\right] = E\left[E\left[e^{tZ}|Y\right]\right]$.

14. A hen lays N eggs, where N has the Poisson distribution with parameter λ. Each egg hatches with probability p independently of the other eggs. Let K be the number of chicks.

(a) Find the expected number of chicks given that $N = n$.

(b) Find the expected number of chicks.

(c) Find the expected number of eggs given that $K = k$.