University College London

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

MSc VIVE

COURSE CODE	:	**MMAI**
TITLE OF EXAMINATION	:	**Mathematical Methods, Algorithms and Implementations**
DATE	:	**7-MAY-1999**
TIME	:	**14.30-17.00**
TIME ALLOWED	:	**2.30 hours**

TURN OVER

Answer 3 questions, at least one from each of Part A and Part B.

Part A *Answer at least one question from this part.*

1) a) Write down definitions for the forward $\mathcal{F}$ and inverse $\mathcal{F}^{-1}$ Fourier Transform (FT) of a continuous function f(t)

[4 Marks]

b) Using the definitions of part a), show that if the FT of a function h(t) is H(w), then the FT of h(kt), where k is a positive constant, is given by

$$\mathcal{F}[h(kt)] = (1/k)\ H(w/k)$$

[4 Marks]

c) Consider the function

$$\mathrm{Rect}_T(t) \quad = \quad \begin{cases} 1 & \text{if } -T < t < T \\ 0 & \text{otherwise} \end{cases}$$

and the function

$$\mathrm{NRect}_T(t) \quad = \quad 1/(2T)\ \mathrm{Rect}_T(t)$$

Find the Fourier Transform of $\mathrm{Rect}_T(t)$

[2 Marks]

d) Show that

$$\mathrm{Rect}_T(kt) \quad = \quad \mathrm{Rect}_{T/k}(t)$$

and sketch diagrams to show the relative shapes of the functions $\mathrm{Rect}_T(kt)$ and $\mathrm{NRect}_T(kt)$, and their Fourier Transforms, as k is

i) doubled

ii) halved

[8 Marks]

e) What generalised function results as the limit of $\mathrm{NRect}_T(kt)$ as $k \to \infty$ and what is its Fourier Transform ?

[3 Marks]

f) Let g(t) be the generalised function derived in part e). Show that convolution of g(t) with any function f(t) leaves f(t) unchanged. Discuss the relationship of this process to sampling f(t).

[4 Marks]

[TURN OVER]

2) a) A signal $f(t)$ is sampled to form a list $\{f_k;k=0\rightarrow N\}$ with $f_0=f(-T/2)$ and $f_N=f(T/2)$. State the range and sampling interval of the frequencies that are represented in these data.

[2 Marks]

b) Consider the set of samples $\{f_k;k=0\rightarrow N-1\}$ from the sampling process described in part a), with the last sample removed, and let $\{F_k;k=0\rightarrow N-1\}$ be its Discrete Fourier Transform (DFT). Assume that the frequencies in $\{F_k\}$ are ordered from lowest to highest.

If N zeros are added at the beginning and at the end of the list $\{F_k\}$ to form another list $\{G_k;k = 0\rightarrow 3N-1\}$, state which frequencies these represent.

[4 Marks]

c) If the expanded list $\{G_k;k = 0\rightarrow 3N-1\}$ from part b) is passed to an Inverse Discrete Fourier Transform (IDFT) resulting in a list $\{g_k;k = 0\rightarrow 3N-1\}$, state the sampling points in the time domain that this list represents.

[4 Marks]

d) Show that the list $\{g_k;k = 0\rightarrow 3N-1\}$ can also be obtained by convolution with a continuous function $h(t)$ in the time domain. State the form of $h(t)$ and what assumptions are made about the original function $f(t)$. What are the relative advantages and differences between this method and that of part c) ?

[8 Marks]

e) By means of mathematical arguments, or diagrams, or both, compare the effect of interpolating by the function $h(t)$ in part d), and

i) nearest neighbour interpolation

ii) linear interpolation

Why would either of these schemes be used rather than the "ideal" interpolant $h(t)$?

[7 Marks]

[CONTINUED]

3) The following Partial Differential Equation can be used to describe variable conductivity diffusion in two dimensions :

$$\nabla \bullet \left(k(x,y) \nabla f(x,y) \right) = \frac{\partial f(x,y)}{\partial t} \qquad \mathbf{1}$$

a) Consider a finite square solution domain of size d, discretised into a grid with N points on each side. Derive the spatial finite differencing operators for the terms on the left hand side of Equation 1, and therefore formulate the problem as a finite matrix equation :

$$A\,\underline{f} = \frac{\partial \underline{f}}{\partial t} \qquad \mathbf{2}$$

[15 marks]

b) Show how the time-derivative can be discretised to form either an Implicit, Explicit or Alternating Direction Implicit scheme. Consider the relative merits of these different approaches.

[10 marks]

[TURN OVER]

Part B *Answer at least one question from this part.*

4) The company you work for has decided to increase the ability of its Internet telephone to use a compression algorithm that offers better than telephone quality speech. Your previous report identified a software implementation of sub-band ADPCM as offering a good compromise between bandwidth consumption and the companies desire to make the system available to a wide range of users and compatible across a wide range of host platforms. You have now been asked to write a second report, which describes expected software implementation, and which must include:-

a) Background information on the operation of ADPCM compression algorithms, including a block diagram

[8 marks]

b) Identifies where in the block diagram drawn for part a) protection mechanisms against the effects of finite word length arithmetic might be included, which quantisation or overflow characteristic they use, and why

[11 marks]

c) Discusses why extra software should be written to allow the program to benefit from MMX capabilities available on some processors, and identifies which parts of the compression algorithm would substantially benefit from being written using the MMX instruction set.

[6 marks]

5)

a) Standard A4 paper documents of approximately 30cm x 21cm are often produced by laser printing at a resolution of approximately 500 dots/inch (or, equivalently, approximately 200 dots/cm). A design study is to be made into the feasibility of developing an electronic document imaging system, based as far as possible on standard, mass produced solid state camera and computer components, that can achieve the same resolution.

(i) Estimate the size of the image of a single A4 page at this resolution per pixel, and show that it is far larger than the size of standard CCD video cameras.

[Question 5 cont. over page]

[Question 5) a) cont.]

The idea is therefore to image each document page by forming a mosaic of sub-images taken as a camera is scanned over the page. In order to register the sub-images with each other it is essential that there is at least 10% overlap between neighbours.

(ii) Determine the size of the mosaic required and best way to orient the camera relative to the page if a standard 768 x 576 resolution monochrome CCD camera is to be used.

The camera operates at the standard rate of 25 frames/sec and quantises pixels at 8 bits. Calculate in addition:

(iii) the data rate of this camera,

(iv) the amount of data/document page,

(v) the time required to scan each page.

Comment on the implications of (iii), (iv) and (v) for computer memory buffer and disc memory requirements.

[10 marks]

b) Registration of neighbouring sub-images is to be carried out by finding the best match over a range of 25 x 25 pixel displacements of the overlapping areas using a minimum absolute difference (MAD) metric. Show that, in terms of the number of numerical (multiply/add) operations per second, the computation rate required is ~ 1Gops/sec, if the best match is found by exhaustive search., and pages are to be scanned continuously. Comment on how the efficiency might be improved whilst maintaining the exhaustive search.

[5 marks]

c) A particular computer manufacturer is developing a new, low-cost 500 Mhz processor chip that should be capable of effectively carrying out one numerical multiply/add operation every four clock cycles. Discuss how you would use one or more of these processors to meet the computational requirements implied by (b) above. If you propose to use several processors, describe the architecture of the multi-processor system that you think would be most appropriate and explain why you have chosen it.

[10 marks]

[TURN OVER]

6)

a) Describe what is meant by a ‘Gaussian image pyramid’ and explain why it is essential to include low-pass filtering stages in the computation of such a data structure.

In a particular implementation, it is proposed to use a Gaussian low-pass filter with a half-width σ pixels as an FIR filter defined over a half width $w = 3\sigma$ pixels.

(i) Show that the Gaussian filter is separable and symmetric and estimate the computational complexity of implementing it as an FIR filter of width $2w+1$ pixels in terms of the number of numerical operations required/output pixel. Ignore border effects.

(ii) Explain how such filters may be cascaded to produce a Gaussian pyramid whose linear resolution is halved at each level.

(iii) Show that the computational complexity for producing the first five levels of reduced resolution of such a pyramid from an N x M image is approximately $6.8NM(3w+1)$.

[9 marks]

b) Describe how a Gaussian pyramid may be used to reduce the computational load of a cross-correlation template matching algorithm. Illustrate your answer by using the Gaussian pyramid constructed in (a) above to reduce the calculations required to find the best match of a P x Q template in an M x N image, by showing:

(i) that $2\dfrac{NM}{4^5}\dfrac{PQ}{4^5}$ arithmetic operations are required at the coarsest scale,

(ii) that the overall complexity of the template matching is reduced to approximately $2PQ\left(\dfrac{2^4}{3}+\dfrac{NM}{2^{20}}\right)$ arithmetic operations. Border effects may again be ignored.

[10 marks]

c) The above procedures are to be used for substituting the face of a competitor in a live TV game show by a picture of a celebrity. Explain why this is a real-time computation, and estimate the computation rate required for each of (a) and (b) above if the competitor’s face occupies a region of approximately 128 pixels square in images that are 576 x 768 pixels resolution. Assume that the pyramid is constructed by filtering with a Gaussian of half-width $\sigma = \sqrt{2}$ between levels.

[6 marks]

[END OF PAPER]