Exam Questions for B12

March 29, 2000

Answer THREE questions.

If you answer MORE THAN THREE questions only the best three will be counted.

1. This question is about propositional logic.

   a. Explain what is meant by *disjunctive normal form*.

   [4 marks]

   b. Which of the following formulas are satisfiable?

      1. $(p \to \neg p)$
      2. $\neg(p \to (q \to p))$
      3. $\neg((p \to q) \to p)$
      4. $(((p \wedge q) \to r) \wedge \neg(p \to (q \to r)))$

   [8 marks]

   c. Which of the following formulas are valid?

      1. $(p \to \neg p)$
      2. $((p \to q) \to (q \to p))$
      3. $((p \to q) \to (\neg q \to \neg p))$
      4. $(p \to (q \to r)) \to ((p \wedge q) \to r)$.

   [8 marks]

   d. Let $\phi = (((p \vee q) \to p) \to (p \wedge \neg q))$. Construct a complete tableau with ϕ at the root. Use your tableau to determine if ϕ is satisfiable or not. Also, use your tableau to find a formula in disjunctive normal form, equivalent to ϕ.

   [13 marks]

[Total = 33 marks]

2. This question is about boolean algebra.

a. Define the NOR operator, $*$, for boolean algebra, by drawing a truth table for it.

[2 marks]

b. Find an expressions equivalent to the following using the NOR operator only.

1. $-a$
2. $a.b$
3. $a+b$
4. $a.\bar{b}$

[8 marks]

c. Write each of the expressions below in conjunctive normal form, i.e. as a product of sums.

1. $a.b + \bar{a}.\bar{b}$.
2. $\bar{a}.b.c + a.b.\bar{c}$

[10 marks]

d. For each of the following expressions (i) write the expression as a sum of products (disjunctive normal form) (ii) draw a Karnaugh map for the expression and (iii) use your Karnaugh map to simplify the expression, if possible.

1. $(-(\bar{a}.\bar{b}).\bar{b}) + -(\bar{a}+\bar{b})$.
2. $(\bar{a}+\bar{b}+\bar{c}).(a+\bar{b}+\bar{c}).(\bar{a}+b+c).(a+b+c)$.

[13 marks]

[Total = 33 marks]

TURN OVER

3. a. What does it mean when we say that two propositional formulas are *semantically equivalent* to each other?

[5 marks]

b. Let ϕ be an arbitrary propositional formula using any of the connectives $\neg, \vee, \wedge, \rightarrow$. Explain how it is possible to find a propositional formula ϕ^*, semantically equivalent to ϕ, and using only the connectives $\neg, \rightarrow$.

[7 marks]

c. What does it mean when we say that a proof system $\vdash$ is sound and complete for a propositional language?

[8 marks]

Let L be the propositional language of all formulas using only the connectives $\neg$ and $\rightarrow$ and let L^+ be the propositional language of all formulas using only the connectives $\neg, \vee, \wedge, \rightarrow$.

d. Let $\vdash$ be a sound a complete proof system for L using axioms A. Give some additional axiom schemes that can be added to A in order to make the proof system sound and complete for L^+.

[11 marks]

CONTINUED

4. This question is about predicate logic. Let L be a first-order language with predicate symbols $\{Eq^2, Gt^2\}$, function symbols $\{+^2, \times^2\}$ and constant symbols $\{zero, one\}$. Let $Z = (\mathbb{Z}, I)$ be an L-structure where $\mathbb{Z}$ is the set of all integers (positive, zero and negative), and I interprets the symbols as follows.

- $I(Eq^2)(m,n)$ holds if and only if $m = n$
- $I(Gt^2)(m,n)$ holds if and only if $m > n$
- $I(+^2) : (m,n) \mapsto m + n$
- $I(\times^2) : (m,n) \mapsto m \times n$
- $I(zero) = 0, \quad I(one) = 1$

a. Translate the following English sentences into this language.

i. Zero is smaller than one.

ii. There is no least integer.

iii. The product of two positive integers is greater than both integers.

iv. Three is a prime number

v. For any number there is a bigger prime number.

[10 marks]

b. Which of the following formulas are *valid* in Z?

i. $\forall x \exists y Gt^2(y,x)$

ii. $\exists y \forall x Gt^2(y,x)$

iii. $\forall x \forall y (Gt^2(y,x) \leftrightarrow Gt^2(+^2(y, one), x))$

[9 marks]

c. Give an assignment to the free variables to show that the formula

$$Gt^2(y,x) \leftrightarrow \exists z (Gt^2(y,z) \wedge Gt^2(z,x))$$

is *not* valid in Z. Find another L-structure in which the formula *is* valid.

[14 marks]

[Total = 33 marks]

5. a. Explain the difference between *strong induction* and *ordinary induction*.

[4 marks]

b. What do we mean when we say that a set is *countably infinite*?

[5 marks]

c. What is the cardinality of the set of all strings of length n over a binary alphabet $\{0, 1\}$, where n is some natural number.

[4 marks]

d. Prove your answer to the previous question either by induction or strong induction, making it clear which method you are using.

[6 marks]

e. What is the cardinality of the set of all strings of finite length over the alphabet $\{0, 1\}$. Briefly justify your answer.

[7 marks]

f. What is the cardinality of the set of all countably long strings over the alphabet $\{0, 1\}$. Briefly justify your answer.

[7 marks]

[Total = 33 marks]

END OF PAPER