Answer THREE questions, at least ONE from EACH of Sections A and B.

SECTION A

1. a. Consider the following informally stated decision problem:

Instance: A graph G.

Question: Is G connected?

Explain, giving suitable examples, how this informally stated problem can be turned into a formal language.

[12 marks]

b. Define what it means to say that an infinite set is countable.

[2 marks]

c. Let A be a finite alphabet, where $|A| > 0$. Prove that the set of all formal languages over A is not countable.

[12 marks]

d. Give an argument that the set of recursively enumerable languages over A is countably infinite. What does this allow you to conclude.

[7 marks]

[Total 33 marks]

2. a. Let L_1 and L_2 be two languages. Define what it means to say that L_1 reduces to L_2 (in symbols, $L_1 \leq L_2$). Prove that if L_1 is not recursive and $L_1 \leq L_2$ then L_2 is not recursive.

[12 marks]

b. Discuss, including proofs where necessary, what can be concluded if $L_1 \leq L_2$ and L_1 is recursive.

[6 marks]

c. A relation R is reflexive if for all x we have xRx. It is transitive if whenever xRy and yRz we also have xRz. Prove that $\leq$ is reflexive and transitive.

[6 marks]

d. A relation R is symmetric if whenever we have xRy we also have yRx. Prove that $\leq$ is not symmetric.

[6 marks]

e. Let L be a finite language. What can be said about whether L is recursive or otherwise, in the absence of further information?

[3 marks]

[Total 33 marks]

3. a. Define what it means to say that a language L is recursive, and what it means to say that a language L is recursively enumerable. Why is it necessary to make a distinction between these two concepts?

[6 marks]

b. Prove that any recursive language is recursively enumerable.

[2 marks]

c. Define the Halting Problem for Turing Machines and prove that it is not recursive.

[10 marks]

d. Prove, by reducing the empty tape halting problem or otherwise, that the following decision problem is unsolvable.

Instance: Two Turing machines, M_1 and M_2 and an input alphabet T common to both of them.

Question: Let $L(M_1)$ and $L(M_2)$ be the languages accepted by M_1 and M_2. Is it the case that $|L(M_1) \cup L(M_2)| > 2$?

[15 marks]

[Total 33 marks]

SECTION B

4. a. Let A and B be decision problems and suppose $A \in \mathbf{P}$. Suppose further that B has at least one yes-instance and at least one no-instance. Prove that $A \leq B$.

[8 marks]

b. Define the *2-satisfiability problem* 2SAT for propositional logic.

[5 marks]

c. Show that 2SAT is in $\mathbf{P}$.

[12 marks]

d. Assuming that **P** $\neq$ **NP**, does 3SAT reduce in p-time to 2SAT? Give reasons.

[8 marks]

[Total = 33 marks]

5. Let A and B be decision problems.

a. Say what it means for one decision problem to reduce to another in p-time.

[6 marks]

b. What does it mean if we say that A is p*-time equivalent* to B?

[5 marks]

c. Prove that p-time equivalence is *reflexive*, *symmetric* and *transitive*.

[11 marks]

d. If A and B are both **NPC** decision problems, does it follow that A is p-time equivalent to B? Justify your answer.

[11 marks]

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6. a. Define the class of all NP-hard decision problems (but you do not have to define the class **NP**).

[5 marks]

b. Define the *propositional satisfaction problem* (PSAT).

[4 marks]

Consider the following 4-*vertex cover problem* (4VC). An instance is any undirected graph $G = (V, E)$ where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. G is a yes-instance of 4VC if there is a set of vertices $W \subseteq V$ such that

- W 'covers' G, i.e. for each edge $e \in E$ there is a vertex $v \in W$ with $v \in e$.
- $|W| \leq 4$.

c. Prove that 4VC $\leq$ PSAT. For this you should use a propositional language with one proposition p_v for each vertex $v \in V$.

[9 marks]

d. Given that PSAT is NP-complete, and using the reduction in the last question, can you infer anything about the complexity of 4VC? If so, what can you infer?

[6 marks]

e. What is the best complexity class (with respect to p-time reduction) that 4VC can be included in? Briefly justify your answer.

[9 marks]

[Total = 33 marks]

END OF PAPER