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Answer 2 questions

Marks for each part of each question are indicated in square brackets

Calculators are NOT permitted

1. a. Write down definitions for the forward and inverse Discrete Fourier Transform (DFT) between lists $\{f_j\}$ and $\{F_k\}$ of length N.

[4 marks]

b. If the lists $\{f_j\}$ and $\{F_k\}$ are represented as N-dimensional vectors, show that the DFT and its inverse can be represented as $N \times N$ matrices. Give the explicit form for the case $N = 4$ and verify explicitly that the matrix for the Inverse DFT is the inverse of the matrix for the DFT. [You may assume the standard identities $\mathrm{e}^{\mathrm{i}\pi/2} = \mathrm{i}, \mathrm{e}^{\mathrm{i}\pi} = -1, \mathrm{e}^{-\mathrm{i}\pi/2} = -\mathrm{i}, \mathrm{e}^{2\mathrm{i}\pi} = 1$ etc.]

[6 marks]

c. For the $N = 4$ case, show that the list $\{f_0, f_1, f_2, f_3\}$ can be represented by combinations of the basis functions $\{1, \cos(t), \sin(t), \cos(2t)\}$ sampled at a set of discrete time positions $\{t_j\}$ in the interval $[0, 2\pi]$. In your answer, give the set of sample postions $\{t_j\}$ explicitly. Why is $\sin(2t)$ not required as a basis function ?

[6 marks]

d. Show that the coefficients of the sampled basis functions in part c) and the the elements of the list $\{f_0, f_1, f_2, f_3\}$ are related by a 4×4 matrix, and give the inverse of this matrix. Show that these matrices are obtained from the matrices for the DFT and Inverse DFT by linear combinations of rows and columns. Comment on how this applies for general lists of length N.

[9 marks]

[Total 25 marks]

2. a. Define the operation of *convolution* for two continuous one-dimensional functions $f(t), g(t), t \in (-\infty, \infty)$, and for two lists of sampled functions $\{f_j; j = 0 \ldots N-1\}, \{g_j; j = 0 \ldots M-1\}$, and state the convolution theorem in each case.

[5 marks]

b. Suppose you are given a list $\{f_j; j = 0 \ldots N-1\}$ of samples of a function $f(t)$ taken at equal intervals $\{t_j = -T + j2T/N; j = 0 \ldots N-1\}$. This list is passed to a Discrete Fourier Transform (DFT) to produce another list $\{F_j; j = 0 \ldots N-1\}$, which is considered as samples of a continuous function $F(\omega)$. State the range and sampling interval of the parameter ω.

[4 marks]

c. If $g(t;\sigma) = (2\pi\sigma^2)^{-1/2}\exp(-t^2/(2\sigma^2))$ is a Gaussian of width σ, give the form of its Fourier Transform $G(\omega;\sigma)$. If the convolution of $f(t)$ with $g(t;\sigma)$ is denoted $h(t;\sigma)$ then show how to find samples $\{h_j\}$ of $h(t;\sigma)$ at the sample points $\{t_j = -T + j2T/N; j = 0 \ldots N-1\}$ by multiplication of the list $\{F_j\}$ with another list $\{G_j\}$ followed by an Inverse DFT. Show how to find $\{G_j\}$ by

1. sampling $g(t;\sigma)$ followed by a DFT
2. sampling $G(\omega;\sigma)$

State the sample points in each case. Discuss the advantages and/or disadvantages of the above two methods over each other.

[7 marks]

d. Suppose we now require the derivative of $h(t)$, sampled at $\{t_j\}$. Discuss different strategies for doing this utilising combinations of the following operations

1. difference differentiation of the list $\{h_j\}$
2. difference differentiation of the list $\{g_j\}$
3. sampling the derivative of $g(t;\sigma)$
4. sampling the derivative of $G(\omega;\sigma)$
5. sampling the representation of the Fourier Transform of the derivative operator.

What are the relative merits (if any) of these methods.

[9 marks]

[Total 25 marks]

3. Consider the two-dimensional problem

$$A\frac{\partial^2 u(x,y)}{\partial x^2} + 2B\frac{\partial^2 u(x,y)}{\partial x \partial y} + C\frac{\partial^2 u(x,y)}{\partial y^2} = 0 \tag{1}$$

where A, B, C are known constants.

a. By considering solutions of the form

$$u(x,y) = f(ax+by)$$

show that equation 1 can be written in the form

$$\left(Aa^2 + 2Bab + Cb^2\right) f''(ax+by) = 0 \tag{2}$$

[4 marks]

b. Show that equation 2 can be satisfied for any function f by putting

$$b = \lambda a$$

Find the possible values for λ in terms of the original constants A, B, C and show that a general solution to equation 1 has one of the forms

$$u(x,y) = f_1(x+\lambda_1 y) + f_2(x+\lambda_2 y) \tag{3}$$

or

$$u(x,y) = f_1(x+\lambda y) + x f_1(x+\lambda y) \tag{4}$$

[6 marks]

c. Find the values of λ that satisfy equation 2 and verify the solution given by equation (3) for the two cases

1. Wave Equation : $A = 1, B = 0, C = -1/c^2$
2. Laplace Equation : $A = 1, B = 0, C = 1$

[5 marks]

d. Equation 1 is augmented with the boundary conditions

$$u(x,0) = \phi(x)\ , \quad \frac{\partial u(x,0)}{\partial y} = 0$$

For both the wave equation and the Laplace equation show that the solution is

$$u(x,y) = \frac{1}{2}\left(\phi(x + \lambda_1 y) + \phi(x + \lambda_2 y)\right)$$

Sketch the form of the solution for the particular case $\phi(x) = \cos(x)$ for both the wave equation and the Laplace equation and comment on the differences.

[10 marks]

[Total 25 marks]

4. a. Let $\mathcal{L} : \mathrm{X} \to \mathrm{X}$ be a linear operator mapping function from a space X to itself, where X is defined as the space of smooth integrable functions on the interval $[a, b]$. Define the requirements for $\mathcal{L}$ to be self-adjoint

[3 marks]

b. Assuming $\mathcal{L}$ is defined as in part a) and is self-adjoint, consider the eigenvalue problem

$$\mathcal{L}u_j(x) = \lambda_j u_j(x)$$

Show that

1. all λ_j are real
2. If $\lambda_j \neq \lambda_k$ then

$$\int_a^b \overline{u_j}(x)u_k(x)\mathrm{d}x = 0$$

where $\overline{u}$ is the complex conjugate of u.

[4 marks]

c. Consider the case where $\mathcal{L} := \frac{\mathrm{d}^2}{\mathrm{d}x^2}$ and two cases of boundary conditions

1. $u(a) = u(b) = 0$
2. $u'(a) = u'(b) = 0$

In both cases, show that the self-adjoint property is satisfied and determine the set of eigenvalues λ_j and eigenfunctions $u_j(x)$. Verify the properties stated in part b).

[8 marks]

d. Consider a discretised system formed by choosing a grid of points

$$\left\{ x_n = a + n\frac{(b-a)}{N}; n = 0 \ldots N \right\}$$

Show that the finite difference method applied to the operator in part c) leads to an eigenvector problem

$$\mathsf{A}\boldsymbol{u_j} = \lambda_j \boldsymbol{u_j}$$

and give the explicit form of A for the two boundary condition cases given in part c). Pay attention in your answer to the differences in dimension of A. Comment on any difference in solution methods required for the different matrices for the two cases.

[10 marks]

[Total 25 marks]

END OF PAPER