University College London

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

M.Res

COURSE CODE	:	**MMAI**
TITLE OF EXAMINATION	:	**Mathematical Methods, Algorithms and Implementations**
DATE	:	**28-February-2000**
TIME	:	**09.30**
TIME ALLOWED	:	**2 hours**

TURN OVER

Answer two questions

1) a) Assume that a function h(t) is band-limited with maximum frequency ω_c. If it is sampled over a period T, what is the minimum number of samples N_{min} to ensure that no aliasing occurs ? [5 marks]

b) Assuming that a sampled list $\{h_j\}$ $j=1 \rightarrow N_{min}$ has been obtained, describe the steps required to carry out Fourier interpolation to a list of length M that corresponds to taking M samples of h(t) in the sample period T.

[10 marks]

c) Prove that the resultant list in part b) corresponds to the Sinc-interpolated values of the original list.

[10 marks]

2) a) What is meant by a *generalised function* ?

[3 Marks]

b) Give a definition of a *delta-function*, $\delta(t)$, in terms of the limit of a set of ordinary functions, and show that it satisfies the requirements to be a generalised function. State the Fourier Transform of $\delta(t)$.

[5 Marks]

c) From the definition in part b), prove the following properties of a delta-function:

i) *sampling* : $$\int_{-\infty}^{\infty} \delta(t) f(t) dt = f(0)$$

ii) *shifting* : $$\int_{-\infty}^{\infty} \delta(t-a) f(t) dt = f(a)$$

iii) *scaling* : $$\delta(at) = \frac{\delta(t)}{|a|}$$

[6 Marks]

d) Show that if $F(\omega)$ is the Fourier Transform of $f(t)$ then $(i\omega F)$ is the Fourier Transform of $f'(t)$

[3 Marks]

e) Let $\varepsilon(t)$ be the Inverse Fourier Transform of $(i\omega)$. Use the answers to b), c) and d) to deduce the result of the convolution $\varepsilon(t)*f(t)$. Comment on the relationship between this result and the design of convolution filter for edge-detection in image processing

[8 Marks]

3) a) Consider the problem

$$u''(x) = K\, u(x)$$

with $u(0) = u(L) = 0$

show that $u_n(x) = \mathrm{Sin}(n \pi x/L)$ are solutions to this problem for any positive integer n.

[4 Marks]

b) How can the answer to part a) be used to find the solution to the problem:

$$u''(x) - \alpha\, u(x) = f(x)$$

where f(x) is given and $u(0) = u(L) = 0$.

What restrictions on α are necessary for the solution to exist ?

[6 Marks]

c) Show that the solution to part b) can be considered as the convolution of a function G(x,x') with f(x), and find the function G(x,x') as a sum of trigonometric terms.

[11 Marks]

d) Show that $G(x,x_0)$ is the solution to the problem in part b) for the case where f(x) is a delta-function at the point $x = x_0$.

[4 Marks]

4) The following Partial Differential Equation can be used to describe viscid fluid flow in two dimension :

$$\alpha\nabla\left(\nabla\cdot\boldsymbol{f}(x,y)\right) + \beta\nabla^2\boldsymbol{f}(x,y) = \frac{\partial\boldsymbol{f}(x,y)}{\partial t} \qquad \textbf{(1)}$$

where α, β are constants and $\boldsymbol{f} = \begin{pmatrix} u \\ v \end{pmatrix}$ is the velocity vector at every point in a 2D plane.

a) Consider a finite square solution domain of size d, discretised into a grid with N points on each side. Derive the spatial finite differencing operators for each of the terms on the left hand side of Equation (3), and therefore formulate the problem as a finite matrix equation :

$$\mathrm{A}\underline{f} = \frac{\partial\underline{f}}{\partial t} \qquad \textbf{(2)}$$

where $\underline{f} = \begin{pmatrix} \cdot \\ \cdot \\ u_{i,j} \\ v_{i,j} \\ \cdot \\ \cdot \end{pmatrix}$ [17 marks]

b) Show how the time-derivative can be discretised to form either an Implicit or Explicit scheme. Consider the relative merits of these approaches.

[8 marks]

[**END OF PAPER**]