University College London

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

M.Res

COURSE CODE	:	**MMAI**
TITLE OF EXAMINATION	:	**Mathematical Methods, Algorithms and Implementations**
DATE	:	**1-March-1999**
TIME	:	**09.30-11.30**
TIME ALLOWED	:	**2 hours**

TURN OVER

Answer two questions

1) a) Write down definitions for the forward and inverse Fourier Transform (FT) of a continuous function f(t), and the forward and inverse Discrete Fourier Transform (DFT) of a list $\{f_j\}$. What is the Fourier Series of a function, and what are the assumptions about the function such that the series exist ?

[5 Marks]

b) If F(ω) is the Fourier Transform of f(t), state the Fourier Transform of the *shifted function* f(t+a).

[2 Marks]

c) A *periodic function* f(t) is defined with the property

$$f(t + n\,T) = f(t) \qquad \text{for all integer } n \in (-\infty,\infty)$$

Use the result of part b) to show that the Fourier Transform of f(t) is zero except for a discrete set of points $\{F_k\}$ equally spaced at intervals of $2\pi/T$.
[Hint : use the fact that $e^{2\pi i}=1$]

[4 Marks]

d) The periodic function f(t) can also be considered as the convolution of a function $f_0(t) = f(t)\,Rect_{T/2}(t)$ with a Comb function $Comb_T(t)$. By considering the Fourier Transform of $f_0(t)$, show that the values of the coefficients in its Fourier series are identical to samples of the continuous Fourier Transform F(ω) at the points $\{\omega_k = 2\pi k/T\}$, scaled by 1/T.

[7 Marks]

e) By considering now that f(t) is sampled at a set of discrete points $\{t_1, t_2,..t_N\}$ equally spaced with interval ΔT, show that its Fourier Transform is periodic with period 2πN/T, and is given by the Discrete Fourier Transform.

[7 Marks]

[**CONTINUED**]

2) a) A signal f(t) is sampled from -T/2 to T/2 in N samples. State the range and sampling interval of the frequencies that are represented in this data.

[2 Marks]

b) Suppose f(t) is a Gaussian $G(t) = Exp(-t^2/(2\ s^2))$. If T = 4s and N = 6, use diagrams or approximate numerical arguments, or both to estimate the fraction of signal outside the sampling interval in both the temporal and Fourier domains. Show how your answer is changed if

i) T is doubled and N remains constant

ii) T remains constant and N is doubled

iii) Both T and N are doubled

[8 Marks]

c) By use of a diagram, or otherwise, show that the frequencies lying outside the sampling interval in the Fourier domain corrupt the low frequencies in the sampling data (*aliasing)*. Show that a low pass filter applied to the signal before sampling can be used to remove aliasing. Why is an ideal low-pass filter difficult to apply ?

[5 Marks]

d) Suppose that the degree of aliasing is defined as $C = A_{out}/A_{in}$ where A_{out} is the area outside the sampling interval in the Fourier domain, and A_{in} is the area inside. It is proposed to use a Gaussian to prefilter a signal before sampling. Find an expression for s in terms of N and T, that would be appropriate to ensure a reduction in C by a factor of 100.

[10 Marks]

3 a) Show that if h(t) = f(t)*g(t) then

$$\frac{\mathrm{d}h(t)}{\mathrm{d}t} = \frac{\mathrm{d}f(t)}{\mathrm{d}t} * g(t) = f(t) * \frac{\mathrm{d}g(t)}{\mathrm{d}t}$$

where '*' represents convolution.

[4 marks]

b) If a Gaussian convolution kernel of scale σ in two-dimensions is defined as :

$$G(\sigma;x,y) = \frac{1}{2\pi\sigma^2}\mathrm{Exp}\left[-\frac{(x^2+y^2)}{2\sigma^2}\right]$$

find the form of the derivative operators G_x, G_y, G_{xx}, G_{xy}, and G_{yy}, in both the spatial and Fourier domains.

[6 marks]

c) It is proposed to implement a multi-resolution edge detector by looking for local maxima of the *gradient* $|\nabla I_\sigma(x,y)|$ of an image at a given scale, and tracking these across scale from coarse to fine. Describe how to implement this scheme if the gradient at scale σ is implemented as

i) Convolution in the spatial domain

ii) Multiplication in the Fourier domain

Include in your answer a discussion of the relative merits in the two approaches, in terms of computational speed, memory requirements, and accuracy.

[15 marks]

4) a) Consider the problem

$$- u''(x) + k^2 u(x) = 0$$

with boundary conditions

$$u(0) = a \qquad u(L) = b$$

show that $u(x) = A \cos(k x + \phi)$ is a solution to this problem where A and ϕ are constants. By considering the boundary conditions, find expressions for the constants A and ϕ

[10 Marks]

b) Describe in detail how to develop a one dimensional Finite Difference scheme for this problem in the form

$$M \mathbf{u} = \mathbf{q}$$

where M is a tridiagonal matrix, and **u** and **q** are vectors. Include in your description the choice of grid spacing, the dimension of the vectors **u** and **q**, the components of the matrix M and right hand side vector **q**, and appropriate solution schemes for the matrix equation.

[15 Marks]

[**END OF PAPER**]