UNIVERSITY OF LONDON

BA EXAMINATION

for Internal Students

This paper is also taken by Combined Studies Students

PHILOSOPHY

Optional Subject (h): Symbolic Logic

Answer THREE questions, at least ONE from EACH section

SECTION A

1. (i) Present Russell's Paradox. What is its significance for Axiomatic Set Theory?

   (ii) Prove that there is no universal set.

   (iii) Prove that there is is no set of all unit sets.

   (iv) Assuming that there is a set, prove that there is exactly one empty set.

   (v) Prove that every set has a subset.

2. (i) What is (a) a function, (b) an injection, (c) a bijection from A to B, (d) id_A, (e) f^{-1}, where f is an injection, (f) $g \circ f$, where $ran(f) \subseteq dom(g)$?

   Let f be an injection with domain A and range B. Show that

   (ii) f^{-1} is a bijection from B to A.

   (iii) $f^{-1} \circ f = id_A$.

   (iv) Let $x \prec y$ if and only if there is an injection with domain x and range a subset of y. Show that the relation $\prec$ on the class of sets is reflexive, transitive but not symmetric.

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3. (i) Define the sum $\kappa + \lambda$ and the product $\kappa \cdot \lambda$ of pair of cardinals κ, λ. Show that for any cardinals κ, λ, μ $(\kappa + \lambda) \cdot \mu = (\kappa \cdot \mu) + (\lambda \cdot \mu)$.

   (ii) Using "$\times$" for cartesian product, show that $\bigcup\{\{a\} \times D \ : \ a \in A\} = A \times D$.

   (iii) Show that if A and B are sets, so is the class of functions from A to B. Define cardinal exponentiation: μ to the power of λ (i.e. μ^λ). Show that $\mu^\kappa \cdot \mu^\lambda = \mu^{\kappa+\lambda}$ for any cardinals κ, λ, μ.

4. (i) Define: well-ordering; ordinal; successor ordinal; limit ordinal; ω.

   Let " α' " denote the successor of α. Let a set Z be called *inductive* if and only if $\emptyset \in Z$ and, for all ordinals α, if $\alpha \in Z$ then $\alpha' \in Z$.

   (ii) Assuming, for ordinals α and β, that α' is an ordinal and that $\beta \leq \alpha$ if and only if $\beta < \alpha'$, show that ω is an inductive set.

   (iii) Assuming that ω is a transitive set, show that for any inductive set Z, $\omega \subseteq Z$.

5. State the Axiom of Choice. Assuming that the class of ordinals is not a set, use the Axiom of Choice and definition by transfinite recursion on ordinals to prove that every set is equipollent to an ordinal. Hence show that for any set A there is a well-ordering on A.

6. What is it for a set to be Dedekind-infinite?

   (i) Show thate every infinite ordinal is Dedekind-infinite.

   (ii) Assuming that every set is equipollent to some ordinal, show that every infinite set is Dedekind-infinite.

7. (i) What is the sentential modal system T? Show that (a) $P \supset \Diamond P$ is derivable in T and (b) if $P \supset Q$ is derivable in T so is $\Diamond\neg Q \supset \Diamond\neg P$.

   (ii) What is a model of the system T? Show that the axioms for T, TT1 and TT2, are true at any index in any such model, and explain why the rule of necessitation RN is valid with respect to models of T.

   (iii) Show that with respect to models of T

   (a) $\Box P \not\models \Box\Diamond\Box P$

   (b) $\Box P \supset \Box Q \not\models \Box(P \supset Q)$

   (c) $\not\models \Box P \supset \Box\Box P$.

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(iv) Let 'CP' abbreviate 'It is contingently true that P'. Define CP in terms of one of the standard modal operators and explain why $CP \supset \Box CP$ is unacceptable.

(v) What are the model systems S4 and S5? Under what conditions on the accessibility relation on a model M for sentential modal logic is M a model of (a) S4, (b) S5?
Show that $\Diamond\Box P \supset \Box P$ is true at every index of any S5 model but false at some index of some S4 model.

8. (i) What is a model for quantified modal logic that allows domains to vary over indices? With respect to such models specify the semantic rules (i.e. clauses in the truth-definition) for atomic sentences, quantified sentences, and modal sentences. Let a one-place predicate $E^*(x)$ be defined thus: $E^*(x) \equiv \exists y(y = x)$. Describe a model in which $\exists x \Diamond \neg E^*(x)$ is true at @.

(ii) Show that the Barcan formula $\forall x \Box Px \supset \Box \forall x Px$ (a) is false at @ in some model of S4 satisfying the inclusion requirement that if $wRw', Dw \subseteq Dw'$ but (b) is true at @ in every model of S5 satisfying the inclusion requirement.

(iii) Show that the converse of the Barcan formula, namely $\Box \forall x Px \supset \forall x \Box Px$, is true at @ in every model for quantified modal logic satisfying the inclusion requirement. Describe a model in which the converse of the Barcan formula is not true at @.

(iv) When the operator '$\Box$' is interpreted as 'necessarily' in the metaphysical sense, should we accept (a) the Barcan formula, and (b) its converse? Justify your answers, making clear what they imply for constraints on variation of domains over possible worlds.

9. (i) Specify a model of quantified S5 in which $\Box \exists x Px \supset \exists x \Box Px$ is false at @.

(ii) Give the semantic rules for the interpretation of (a) $\lambda x_1 \ldots x_n P(x_1 \ldots x_n)$ and (b) $\iota x Px$, with respect to a model M of quantified modal logic. Specify a model for quantified modal logic in which $\Box P(\iota x Px)$ is true at @ while $\lambda y \Box Py[\iota x Px]$ is not. Say whether $\Box \lambda y Py[\iota x Px]$ is true at @ in that model and justify your answer.

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(iii) Assess briefly the following.

Mathematicians may conceivably be said to be necessarily rational and not necessarily two-legged; and cyclists necessarily two-legged and not necessarily rational. But what of an individual who counts among his eccentricities both mathematics and cycling? Is this concrete individual necessarily rational and contingently two-legged, or vice versa? Just insofar as we are talking referentially of the object ... there is no semblance of sense in rating some of his attributes as necessary and others contingent.

From Quine, *Word and Object.*

10. Let '$\mathbf{A}p$' (for 'Actually P') be interpreted according to the rule: $\mathbf{A}P$ is true at w in M if and only P is true at @ in M.

   (i) Show that there is a model for quantified modal logic in which $\Box\exists xPx$ is false at @ and $\Box\exists x\mathbf{A}Px$ is true at @.

   (ii) Show that there is a model M for quantified modal logic with the same domain at every index, such that for some index w other than @ the value ιxPx at w in M differs from the value $\iota x\mathbf{A}Px$ at w in M.

   (iii) Consider models for quantified modal logic with the same domain at every index, such that $\exists x(Px \mathbin{\&} \forall y(Py \supset x = y))$ is true at every index. Answer with justification the following:

   (a) Is $\Box(\iota xPx = \iota xPx)$ true at @ in every such model?

   (b) Is $\Box(\iota xPx = \iota x\mathbf{A}Px)$ true at @ in every such model?

   (c) Under what circumstances is ιxPx a rigid designator in such a model? If ιxPx is a rigid designator in such a model, is $\Box(\iota xPx = \iota x\mathbf{A}Px)$ true at @ in that model?

   (iv) Formalise the following in the language of quantified modal logic with the 'actually' operator:

   (a) There could have been a giraffe taller than the actual tallest giraffe.

   (b) The actual height of the Eiffel Tower.

   (c) The Eiffel Tower could have been taller than it actually is.

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SECTION B

11. The $\vee$-language is the smallest set V containing all propositional symbols and such that $\phi \vee \psi \in V$ if $\phi, \psi \in V$.
    — The $\vee$-calculus for the $\vee$-language has CUT plus the following rules:

    a. $\alpha \vdash_\vee \alpha \vee \beta$
       $\beta \vdash_\vee \alpha \vee \beta$
    b. If $\alpha \vdash_\vee \gamma$ and $\beta \vdash_\vee \gamma$ then $\alpha \vee \beta \vdash_\vee \gamma$.

    So $\Sigma \vdash_\vee \phi$ means that there is a deduction of ϕ from assumptions in Σ using only cut and the above rules.

    (i) Show that (b) holds in Procal, i.e., If $\alpha \vdash_0 \gamma$ and $\beta \vdash_0 \gamma$ then $\alpha \vee \beta \vdash_0 \gamma$.
    You may use the fact that $\vdash_0 (\neg\delta \to \xi) \to ((\delta \to \xi) \to \xi)$.

    — A $\vee$-valuation is a mapping σ assigning to every propositional symbol P a value P^σ from $\{\top, \bot\}$ which assigns $\phi \vee \psi$ the value $\bot$ if it assigns $\bot$ to both disjuncts, and assigns $\phi \vee \psi$ the value $\top$ in all other cases.
    — A $\vee$-formula α is a $\vee$-consequence of the set Σ of $\vee$-formulas, notation $\Sigma \models_\vee \alpha$, if every $\vee$-valuations which assigns $\top$ to all formulas in Σ, assigns $\top$ to α.

    (ii) Let Γ be a set if $\vee$-formulas which is maximal with respect to the fact that $\Gamma \nvdash_\vee \alpha$ for some fixed α. I.e., if β is a $\vee$-formula and $\beta \notin \Gamma$, then $\Gamma \cup \{\beta\} \vdash_\vee \alpha$.
    Given the $\vee$-valuation σ satisfying $P^\sigma = \top \iff p \in \Gamma$ for all propositional symbols P. Prove that for all $\vee$-formulas ϕ :$\phi^\sigma = \top \iff \phi \in \Gamma$.

    (iii) Given the fact that every $\Sigma \nvdash_\vee \alpha$ is included in a set Γ which is maximal with respect to this fact, show that the $\vee$-calculus is complete with respect to $\vee$-valuations. That is, if $\Sigma \models_\vee \alpha$ then $\Sigma \vdash_\vee \alpha$

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12. Show the following facts, using the basic semantic definition.

    (i) $\alpha \rightarrow \exists x\beta$ and $\exists x(\alpha \rightarrow \beta)$ are logically equivalent if x does not occur free in α.

    (ii) $\exists x(Ax \rightarrow \forall x Ax)$ is true in all $\mathcal{L}$-structures.

    (iii) $\exists x(\exists x Ax \rightarrow Ax)$ is true in all $\mathcal{L}$-structures.

13. Let $\vdash^*$ be the relation of deducibility in a calculus that has CUT, the Inconsistency Effect, the Deduction theorem and Reductio.

    (i) Show that $\neg\neg\alpha \rightarrow \alpha \vdash^* ((\alpha \rightarrow \beta) \rightarrow \alpha) \rightarrow \alpha$.

    (ii) show that, $\neg\neg\alpha \rightarrow \alpha, \beta \rightarrow \alpha, (\beta \rightarrow \gamma) \rightarrow \alpha \vdash^* \alpha$.

14. (i) For Σ a set of propositional logical sentences it holds that "if $\Sigma \models \phi$, then there is some finite $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models \phi$". Use this to prove the compactness theorem: if Σ is a set of formulas such that every finite subset is satisfiable, then Σ is satisfiable.

    (ii) Assume that every finite subset of Σ is satisfiable. Show that, for any propositional sentence α, the same holds for at least one of $\Sigma \cup \{\alpha\}$ and $\Sigma \cup \{\neg\alpha\}$.

    (iii) Let Δ be a set of propositional formulas such that every finite subset of Δ is satisfiable and for every sentence ϕ, $\phi \in \Delta$ or $\neg\phi \in \Delta$. Define the truth valuation σ by $P^\sigma = \top$ if $P \in \Delta$ and $P^\sigma = \bot$ if $P \notin \Delta$ for each propositional variable P. Show that for every sentence ϕ we have: $\phi^\sigma = \top$ iff $\phi \in \Delta$.

15. (i) State the soundness theorem for propositional logic and show how this theorem entails that Propcal is consistent.

    (ii) State the completeness theorem for the propositional calculus and show how it follows from the consistency lemma: "every consistent set is satisfiable."

    (iii) Prove the compactness theorem: if Φ is a set of formulas such that every finite subset of Φ is satisfiable, then Φ is satisfiable.

    You may assume the soundness and completeness of the propositional calculus.

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16. Let $\mathcal{L}$ be the first order language of arithmetic and let the binary predicate letter '<' be defined by $t < s \equiv_{df} \exists y(y \neq 0 \ \wedge \ t + y = s)$. Consider the set

$$\Sigma_1 = \{\forall x \neg(x < x), \forall x \forall y \forall z(x < y \wedge y < z \rightarrow x < z)\}$$

(i) Show, by giving a deduction, that $\Sigma_1 \vdash \forall x \forall y(x < y \rightarrow \neg y < x)$ (Hint: show that $\Sigma_1 \cup \{(a < b) \wedge (b < a)\} \vdash$ for a, b not occurring in Σ and proceed from there.)

(ii) Let $\Sigma_2 = \Sigma_1 \cup \{\forall x \exists y(x < y)\}$
Argue that Σ_2 has no finite models.

17. Suppose S is the set

$$S = \{\mathbf{s}_n + \mathbf{s}_m = \mathbf{s}_{m+n} \mid n, m \in N\} \cup \{\mathbf{s}_n \times \mathbf{s}_m = \mathbf{s}_{n \times m} \mid n, m \in N\}$$

Let Γ be a complete and recursively decidable theory in the language of first-order arithmetic such that $S \subseteq \Gamma$. Prove that S is not a set of postulates for Γ.

18. Define the classes of

(i) Arithmetic relations (A)

(ii) Elementary relations (E)

(iii) Recursive relations (R)

(iv) Recursively Enumerable relations (RE)

Formulate relations between the classes A and E, E and R, R and RE, and RE and A.

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19. (i) Let Γ be a recursive set of sentences. Argue that the relation $\mathrm{Ded}_\Gamma(x, y)$ given by

    $Ded_\Gamma(x, y) \iff x$ is a the code of a sentence ϕ and y is the code of sequence-of-sentences that constitutes a deduction of ϕ from Γ,

    is recursive.

    (ii) Outline a proof that a relation P is recursively enumerable if and only if it is weakly representable in Π_0.

20. (i) Suppose that $R(\mathfrak{x}, y)$ is an $n+1$-ary recursive relation. Show that applying *bounded* universal quantification to R, i.e., $\forall y(y < u \rightarrow R(\mathfrak{x}, y))$, gives again an $n + 1$-ary recursive relation.

    (ii) Let $f\mathfrak{x} = \mu y P(\mathfrak{x}, y)$ be the function which assigns to $\mathfrak{x}$ the minimal y such that $P(\mathfrak{x}, y)$ if there is such a y and gives ∞ otherwise. Show that f is a recursive function if P is a recursive relation. (μy is here the so-called 'minimization' operator).

    (iii) Given that an n-ary relation $R(\mathfrak{x})$ is recursively enumerable if and only if it is of the form $\exists y P(\mathfrak{x}, y)$ for some $n + 1$-ary recursive relation P, show that R is recursive if both R and $\neg R$ are recursively enumerable by constructing a recursive relation using the 'minimization' operator μy. (Hint: the disjunction of recursive relations is itself recursive.)

21. Outline a proof of the First Incompleteness theorem: Given a sound and axiomatic theory Σ we can find a true sentence of the form $\forall \mathbf{x_1} \dots \forall \mathbf{x_m}(\mathbf{t} \neq \mathbf{r})$ that does not belong to Σ.

22. Outline a proof of the Second Incompleteness Theorem: "Let Σ be an axiomatic theory that includes First-order Peano Arithmetic. If Σ is consistent, then the sentence $\mathbf{CONSIS}_\Sigma$, which expresses this fact formally, is not in Σ."

END OF PAPER