UNIVERSITY OF SURREY©

B. Sc. Undergraduate Programmes in Mathematical Studies

Level HE3 Examination

Module MS334 ADVANCED STOCHASTIC MODELLING

Time allowed – 2 hrs

Spring Semester 2008

Attempt THREE questions
If a candidate attempts more than THREE questions only the best THREE questions will be taken into account.

There is one Handout containing three formulas.

Throughout, $B(t)$ denotes standard Brownian motion. Any required properties of Brownian motion may be assumed without proof, but they must be explicitly mentioned when they are used.

Question 1
Consider the standard Brownian motion $B(t)$.

(a) Let $a > 0$. Define the *first hit time* τ_a. [2]

(b) For each $a > 0$, compute the probability distribution of τ_a (any properties of $B(t)$ that you use must be stated explicitly). [7]

(c) Compute the probability distribution of $M(t) = \max_{s \in [0,t]} B(s)$. [3]

(d) Show that reflected Brownian motion $R(t) = |B(t)|$ has the same probability distribution as $M(t)$ for each fixed $t \geq 0$. Hence or otherwise, compute $E(M(t))$. [7]

(e) Prove that $R(t)$ satisfies the Markov property

$$P(R(t) < x | R(t_1) = x_1, \ldots, R(t_k) = x_k) = P(R(t) < x | R(t_k) = x_k),$$

for all $t_1 < t_2 < \cdots < t_k < t$. (You may assume without proof that $B(t)$ satisfies this property.) [3]

(f) Does $M(t)$ satisfy the Markov property? Briefly justify your answer, referring to the figure below if you so wish. [3]

Question 2

(a) Define Brownian motion $X(t)$ with drift parameter μ and variance parameter σ^2. [2]

(b) Let Y be a random variable with a standard normal distribution and let $\lambda \in \mathbb{R}$. Prove that $E(e^{\lambda Y}) = e^{\frac{1}{2}\lambda^2}$. [3]

(c) Let $Z(t) = ze^{X(t)}$ be geometric Brownian motion, where $z > 0$ is fixed. Use part (b) to compute $E(Z(t))$ and $\mathrm{Var}(Z(t))$. Define the drift parameter α for $Z(t)$. [4]

(d) Suppose that $z = 1$, $\alpha = -5$, $\sigma^2 = 9$. Compute $P(Z(4) < 8|Z(1) = 5)$.

(You may express the answer in the form $P(Y < c)$ where $Y \sim N(0, 1)$.) [3]

(e) Define what it means for $X(t)$ to be a *Gaussian process* and define the associated *mean function* $\mu(t)$ and covariance function $\Sigma(s, t)$. [4]

(f) Define $X(t) = B(t) - tB(1)$, $t \geq 0$. Explain why $X(t)$ is a Gaussian process. For $0 < t_1 < t_2 < 1$, determine the distribution of the random vector $\begin{pmatrix} X(t_1) \\ X(t_2) \end{pmatrix}$ and compute the covariance function of $X(t)$.

By now, you should have recognised this Gaussian process. Write down the usual definition of $X(t)$. [9]

Question 3

Let $\{X(t); t \geq 0\}$ be a stochastic process with continuous time and continuous state-space. Denote the joint density functions by $p(x_1, t_1; x_2, t_2; \dots)$ and the conditional density functions by $p(x_1, t_1; x_2, t_2; \dots | y_1, \tau_1; y_2, \tau_2; \dots)$.

(a) State the *Markov property* in terms of the conditional density functions. [2]

(b) Suppose that $X(t)$ is a Markov processes.

(i) Prove that

$$p(x_1, t_1; x_2, t_2; \dots; x_n, t_n) = p(x_1, t_1 | x_2, t_2) p(x_2, t_2 | x_3, t_3) \dots p(x_{n-1}, t_{n-1} | x_n, t_n) p(x_n, t_n),$$

for all $t_1 \geq t_2 \geq \dots \geq t_n$ and all $n \geq 1$. [3]

(ii) Derive the *Chapman-Kolmogorov equation.* [3]

(c) The differential form of the Chapman-Kolmogorov equation can be written in the form

$$\begin{aligned}\frac{\partial}{\partial t} p(z, t | y, t') = & \int \big[W(z | x, t) p(x, t | y, t')) - W(x | z, t) p(z, t | y, t')) \big] dx \\ & - \frac{\partial}{\partial z} [D_1(z, t) p(z, t | y, t')] + \frac{1}{2} \frac{\partial^2}{\partial z^2} [D_2(z, t) p(z, t | y, t')],\end{aligned}$$

for $t \geq t'$ where $D_2 \geq 0$, $W \geq 0$. Do not derive this equation(!) but answer the following:

(i) Write down the initial condition for $p(z, t | y, t)$. [1]

(ii) Which special case is called the *Master equation*? Use this equation and the approximation

$$p(z, t + \Delta t | y, t) \sim p(z, t | y, t) + \frac{\partial}{\partial t} p(z, t | y, t) \Delta t,$$

to explain why the resulting stochastic processes are called *jump processes.* [6]

(iii) Which special case is called the *Fokker-Planck equation*? Prove that the density for $B(t)$ (with $y = 0$, $t' = 0$) satisfies a particularly simple Fokker-Planck equation. [6]

(iv) The Liouville equation is the special case $W \equiv 0$, $D_2 \equiv 0$. Let $x(y, t)$ be the unique solution to the initial value problem

$$\frac{dx}{dt} = D_1(x, t), \quad x(t') = y.$$

Verify that the deterministic process $p(z, t | y, t') = \delta(z - x(y, t))$ satisfies the Liouville equation. [4]

Question 4

(a) Give the definition for $\{S_n; n \geq 1\}$ to be a martingale. [3]

(b) Prove that $E\{E(Y|X)\} = E(Y)$. Hence prove that if $\{S_n; n \geq 1\}$ is a martingale, then $E(S_n) = E(S_1)$ for all $n \geq 1$. [5]

(c) Let $0 < t_0 < t_1 < t_2 < \cdots$, and define $S_n = B(t_n)$. Prove that $\{S_n; n \geq 1\}$ is a martingale. [3]

(d) Let $0 < t_0 < t_1 < t_2 < \cdots$, and define $S_n = B(t_n)^2 - t_n$. Prove that $\{S_n; n \geq 1\}$ is a martingale. [4]

(e) Give examples of martingales of the form $S_n = X_1 + \cdots X_n$ for which

(i) the random variables X_j are not independent.

(ii) the sequence $\{S_n; n \geq 1\}$ does not have the Markov property.

(You may either give one example for which both (i) and (ii) are satisfied, or two separate examples.) [5]

(f) Let $\{S_n; n \geq 1\}$ be a non-negative martingale. Prove that

$$P(S_n \geq \lambda) \leq \frac{E(S_1)}{\lambda},$$

for all $n \geq 1$ and all $\lambda > 0$.

State (but do not prove) the maximal inequality for non-negative martingales. [5]

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