Examination Paper

B.Sc. Undergraduate Programmes in Mathematical Studies

Level HE3 Examination

Module MS334 Advanced Stochastic Modelling

Time allowed – 2 hours | Autumn Semester 2006

Attempt THREE questions. If any candidate attempts more that THREE questions, only the best THREE solutions will be taken into account.

Internal Examiner: M J Kearney

Question 1

In what follows, you may assume that the normal distribution $N(\mu,\sigma^2)$ has the density function

$$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}.$$

(a) List the essential defining properties of standard Brownian motion, $B(t)$.

[8]

(b) Given that $B(5)=4$, calculate the probability that $B(9)<8$, expressing you answer in terms of the distribution of the standard normal variable Y.

[2]

(c) Derive an expression for the joint density function $f_{s,t}(x,y)$ of $B(s)$ and $B(t)$, where $t\geq s$.

[3]

(d) Hence, or otherwise, given that $B(10)=8$, calculate the probability that $B(5)<4$. You may use the fact that $\int_0^\infty e^{-x^2}dx=\frac{\sqrt{\pi}}{2}$.

[5]

(e) Explain what is meant by reflected Brownian motion, $R(t)$, and show why its density function satisfies $f_{R(t)}(x)=2f_{B(t)}(x)$ for $x\geq 0$.

[3]

(f) Prove that if an arbitrary stochastic process $X(t)$ has stationary increments its mean value $\mu(t)=E(X(t))$ must satisfy $\mu(t+s)=\mu(t)+\mu(s)$. Hence explain why reflected Brownian motion $R(t)$ does not have stationary increments.

[4]

Question 2

(a) Define what is meant by absorbed Brownian motion, $A(t)$, and briefly explain for what kinds of problems this is a useful model.

[4]

(b) Prove using the reflection principle that

$$P(A(t) > y | A(0) = x) = P\left(Y > \frac{y-x}{\sqrt{t}}\right) - P\left(Y > \frac{y+x}{\sqrt{t}}\right)$$

where Y is a random variable which has a standard normal distribution.

[10]

(c) Hence show that

$$P(A(t) = 0 | A(0) = x) = 2[1 - P(Y < x/\sqrt{t})].$$

What does this result imply about the behaviour of $A(t)$ as $t \to \infty$?

[3]

(d) Calculate the expected value $E[A(t)]$ and comment on your answer in relation to your findings in part (c).

[8]

Question 3

A certain stochastic process starts at $x(t=0)=y\geq 0$ and evolves according to the Ornstein-Uhlenbeck stochastic differential equation

$$dx = -\gamma x dt + \sigma\, dB(t)$$

where $B(t)$ is a standard Brownian motion. We are interested in the mean time it takes for the process to reach $x=0$ when $\gamma > 0$ is small. One may show that the probability $Q(t,y)$ that the process does <u>not</u> cross the line $x=0$ up to time t obeys the backward Fokker-Planck equation

$$\frac{\partial Q}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 Q}{\partial y^2} - \gamma y \frac{\partial Q}{\partial y}.$$

(a) Explain why the boundary conditions for this equation are $Q(t=0,y)=1$ and $Q(t,y=0)=0$, and also explain why $Q(t=\infty,y)=0$.

[3]

(b) Show that the mean first passage time $T(y)$ for the process to reach $x=0$ satisfies the equation

$$\frac{\sigma^2}{2}\frac{d^2T(y)}{dy^2} - \gamma y \frac{dT(y)}{dy} = -1$$

subject to the boundary condition $T(y=0)=0$.

[8]

(c) Show that the formal solution for $T(y)$ may be written as

$$T(y) = \frac{2}{\sigma^2}\int_0^y \exp(\gamma t^2/\sigma^2)\int_t^\infty \exp(-\gamma s^2/\sigma^2)\,ds\,dt.$$

[8]

(d) Hence show that as $\gamma \to 0$

$$T(y) = \frac{y}{\sigma}\sqrt{\frac{\pi}{\gamma}} - \frac{y^2}{\sigma^2} + O(\sqrt{\gamma}).$$

You may use the fact that $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$.

[6]

Question 4

Let $M(t) = \max\{B(s) : 0 \le s \le t\}$ be the maximum value attained by a standard Brownian motion $B(s)$ on the time interval $[0, t]$.

(a) Explain, using diagrams or otherwise, why the first passage time $T(x)$ for a standard Brownian motion to reach a value $x > 0$ satisfies $P(T(x) < t) = P(M(t) > x)$.

[4]

(b) Discuss in outline form the steps in the argument which show that the probability density of $M(t)$ for $x \ge 0$ is given by

$$f_{M(t)}(x) = \sqrt{\frac{2}{\pi t}} \exp\left\{-\frac{x^2}{2t}\right\}.$$

[10]

(c) Hence show that $P(T(x) < t) = 2[1 - P(Y < x/\sqrt{t})]$, where Y is the standard normal variable.

[3]

(d) You own shares in a company whose price follows a standard Brownian motion. When you purchased the shares their price was P and the present price is Q where $Q < P$. Your strategy is to sell the shares once the price returns to P or after a specified time T passes, whichever happens soonest. Calculate the probability that you will lose money if you follow this strategy.

[8]

Professor M J Kearney