UNIVERSITY OF SURREY©

B. Sc. Undergraduate Programmes in Mathematical Studies
M. Math. Undergraduate Programmes in Mathematical Studies

Level HE3 Examination

Module MS333 STATISTICAL METHODS FOR BUSINESS AND FINANCE

Time allowed – 2 hours | Spring Semester 2008

Attempt THREE questions. If any candidate attempts more than THREE questions only the best THREE solutions will be taken into account.
Students may use approved calculators.
Cambridge Statistical Tables will be provided.

Question 1

(a) A time series is to be smoothed using a weighted moving average filter, obtained by fitting a cubic polynomial to successive groups of 7 observations. Show that, if least squares fitting is used, the filter is

$$\frac{1}{21}(-2, 3, 6, \mathbf{7}).$$

$\Big[$**Note** : It can be assumed that $\displaystyle\sum_{t=-3}^{3} t^2 = 28, \quad \sum_{t=-3}^{3} t^4 = 196.\Big]$ [11]

(b) Consider the time series

$$X_t = m_t + Y_t, \quad (t = 1, 2, \ldots, n),$$

where $m_t = \beta_1\tau_t + \beta_2\tau_t^2$ is an assumed quadratic trend function which is known to pass through the origin and Y_t is a random disturbance with zero mean and variance σ^2. It is required to estimate β_1, β_2 by minimizing the sum of squares

$$S(\beta) = \sum_{t=1}^{n}\left(X_t - \beta_1\tau_t - \beta_2\tau_t^2\right)^2, \qquad \text{where } \beta = \begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix}.$$

Derive the normal equations and, hence or otherwise, show that they can be expressed in the form $G = H\,\widehat{\beta}$, where

$$G = \begin{bmatrix}\sum_{t=1}^{n}\tau_t X_t\\ \sum_{t=1}^{n}\tau_t^2 X_t\end{bmatrix} \qquad \text{and} \quad H = \begin{bmatrix}\sum_{t=1}^{n}\tau_t^2 & \sum_{t=1}^{n}\tau_t^3\\ \sum_{t=1}^{n}\tau_t^3 & \sum_{t=1}^{n}\tau_t^4\end{bmatrix}.$$ [7]

(c) The trend function for an industrial production index X_t is known to pass through the origin and 30 values of X_t are available over an adjusted time-scale τ_t. Summary statistics concerning these data are given in the box below. Assuming that a quadratic function $m_t = \beta_1\tau_t + \beta_2\tau_t^2$ is a reasonable representation of the trend for the industrial production index, estimate the parameters of m_t and comment **briefly** on your results.

Statistic	Value	Statistic	Value	Statistic	Value
$\sum_{t=1}^{30}\tau_t X_t$ =	5.25	$\sum_{t=1}^{30}\tau_t^2 X_t$ =	20.5	$\sum_{t=1}^{30}\tau_t^2$ =	3.8
$\sum_{t=1}^{30}\tau_t^3$ =	14.6	$\sum_{t=1}^{30}\tau_t^4$ =	60.0		

[**Note:** It can be assumed that $\text{Var}(\widehat{\beta}) = \widehat{\sigma}^{\,2}\,\text{H}^{-1}$ where $\widehat{\sigma}^{\,2} = 0.052$.] [7]

Question 2

(a) The time series U_t and Y_t are defined by $U_0 = 0$, $U_t = Z_1^{(1)} + \ldots + Z_t^{(1)}$ and by $Y_t = U_t + Z_t^{(2)}$, where $Z_t^{(1)}$, $Z_t^{(2)}$ are uncorrelated white noise processes with expectations zero and with variances σ_1^2, σ_2^2 respectively, i.e. $WN(0, \sigma_1^2)$ and $WN(0, \sigma_2^2)$ respectively.

(i) Show that $E(U_t) = E(Y_t) = 0$, that $\text{Var}(U_t) = t\sigma_1^2$ and $\text{Var}(Y_t) = t\sigma_1^2 + \sigma_2^2$, and that $\text{Cov}(U_t, U_{t+\tau}) = \text{Cov}(Y_t, Y_{t+\tau}) = t\sigma_1^2$ for all $\tau \geq 1$. Hence deduce that Y_t is not stationary. [6]

(ii) Show that the differenced series $X_t = Y_t - Y_{t-1}$ is stationary and that it has autocorrelation function given by

$$\rho(1) = -\frac{\sigma_2^2}{\sigma_1^2 + 2\sigma_2^2} \quad \text{and } \rho(\tau) = 0 \quad \text{for all } \tau \geq 2.$$

[6]

(b) It is required to find the moving average representation for an ARMA(p, q) model which is specified by

$$X_t - \tfrac{1}{3}X_{t-1} - \tfrac{1}{4}X_{t-2} + \tfrac{1}{12}X_{t-3} = Z_t + \tfrac{1}{2}Z_{t-1},$$

where $\{Z_t\} \sim WN(0, \sigma^2)$.

(i) Find the values of p, q so that there is no parameter redundancy in the model and check whether the model is causal. [5]

(ii) Show that X_t can be represented in operator form by

$$X_t = \left(1 - \alpha_1 B\right)^{-1} \left(1 - \alpha_2 B\right)^{-1} Z_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$$

where B is the backshift operator and α_1, α_2 are suitably defined real numbers which should be specified. [3]

(iii) Confirm that the value of ψ_j takes the form

$$\psi_0 = 1,\ \psi_1 = \alpha_1 + \alpha_2,\ \psi_j = \frac{\alpha_1^{j+1} - \alpha_2^{j+1}}{\alpha_1 - \alpha_2} \quad \text{for } j \geq 2.$$

[5]

Question 3
An ARMA$(1,1)$ process is defined by

$$X_t - \phi\, X_{t-1} = Z_t + \theta\, Z_{t-1},$$

where $\{Z_t\} \sim WN(0, \sigma^2)$ and $\phi + \theta \neq 0$.

(a) State the conditions for the ARMA$(1,1)$ process to be stationary and invertible. [4]

(b) By multiplying the model equation by $X_{t-\tau}$ and taking expectations, or otherwise, obtain the autocovariance equations

$$\begin{aligned} \gamma(0) &= \phi\gamma(1) + \sigma^2(1 + \phi\theta + \theta^2) \\ \gamma(1) &= \phi\gamma(0) + \sigma^2\theta \\ \gamma(\tau) &= \phi\gamma(\tau - 1) \quad (\tau \geq 2). \end{aligned}$$

Hence show that

$$\gamma(0) = \frac{1 + 2\theta\phi + \theta^2}{1 - \phi^2}\,\sigma^2 \quad \text{and} \quad \gamma(\tau) = \frac{(1 + \theta\phi)(\phi + \theta)}{1 - \phi^2}\,\phi^{\tau-1}\sigma^2, \quad \text{for } \tau \geq 1.$$

[9]

(c) It is assumed that the values of the autocorrelation of lag 1,

$$\rho(1) = \frac{(1 + \theta\phi)(\phi + \theta)}{1 + 2\theta\phi + \theta^2},$$

and of the parameter ϕ are known, and it is required to find the value for θ. Let π_0 and π_1 be defined by

$$\pi_0 = 1 + \phi^2 - 2\phi\rho(1) \quad \text{and} \quad \pi_1 = \rho(1) - \phi.$$

Show that $\theta\,\pi_0 = (1 + \theta^2)\,\pi_1$ and hence describe **briefly** how the value of θ can be found. [7]

(d) The values of $\rho(1)$ and ϕ for a stationary and invertible ARMA(1,1) process are found to be $\rho(1) = 11/14$ and $\phi = 2/3$. Using equations from part (c), obtain the value of θ. [5]

Question 4

(a) Give an expression for the autocorrelation function (ACF) $\rho(\tau)$ of lag τ for a moving average (MA(q)) process. Explain **briefly** how the sample ACF $\widehat{\rho}(\tau)$ is useful for helping to identify an ARMA(p, q) process. [4]

(b) Define the Yule-Walker equations for an autoregressive (AR(p)) process and, hence or otherwise, define the partial autocorrelation function (PACF) $\phi_{\tau\tau}$ of lag τ. Explain **briefly** how the sample PACF $\widehat{\phi}_{\tau\tau}$ is useful for helping to identify an ARMA(p, q) process. [6]

(c) Values of the sample ACF and of the sample PACF for lags $\tau = 1, \ldots, 10$ of an observed time series are given in the following tables. Explain **briefly** what kind of model you think that the time series is likely to follow.

τ	1	2	3	4	5	6	7	8	9	10
$\widehat{\rho}(\tau)$	0.91	0.75	0.56	0.29	0.23	0.11	-0.09	0.02	0.01	-0.07

τ	1	2	3	4	5	6	7	8	9	10
$\widehat{\phi}_{\tau\tau}$	0.91	0.07	-0.06	0.03	-0.02	0.02	-0.06	0.01	-0.07	0.03

[4]

(d) Suppose that a time series for n observations is identified as an ARMA(p, q) process, where $p = 1$, $q = 2$, and that the three parameters have been estimated. Define the *residual autocorrelation* of lag τ, $\widehat{\rho}_e(\tau)$, and specify the asymptotic distribution of $\widehat{\rho}_e(\tau)$ for large n if the model has been identified correctly. Use this result to suggest whether or not the time series is identified correctly as an ARMA$(1, 2)$ process, given the following ten values for $\widehat{\rho}_e(\tau)$ from a realization of size $n = 75$.

τ	1	2	3	4	5	6	7	8	9	10
$\widehat{\rho}_e(\tau)$	0.09	-0.15	0.06	0.00	-0.03	-0.10	-0.07	0.18	0.02	0.01

[7]

(e) Apply the Ljung-Box-Pierce portmanteau statistic to test if the time series is identified correctly as an ARMA$(1, 2)$ process. [4]

Question 5

(a) A time series X_t is believed to exhibit a seasonal pattern. Define the terms:

(i) the *seasonal period* s;

(ii) the seasonal difference operator ∇_s. [4]

Hence, or otherwise, specify the model equation for the seasonal integrated ARMA (SARIMA) model, denoted SARIMA$(p, d, q) \times (P, D, Q)_s$, which may be employed to describe the seasonal time series. Describe **briefly** how this model allows for the possibility of a non-seasonal trend component in addition to seasonality. [6]

(b) Consider quarterly data such that observations

$$\{X_{t-i}, X_{t-4-i}, \ldots, X_{t-4j-i}, \ldots\}$$

belong to the ith period, where $i = 0, 1, 2, 3$. Let E_{t-i} be an exponentially weighted moving average of observations in the ith period, i.e.

$$E_{t-i} = (1+\theta)\{X_{t-i} - \theta X_{t-4-i} + \theta^2 X_{t-8-i} + \ldots + (-\theta)^j X_{t-4j-i} + \ldots\},$$

where θ is a parameter restricted by $|\theta| < 1$.

(i) A seasonal ARIMA$(0, 0, 0) \times (0, 1, 1)_4$ model for X_t is given by the equation

$$\nabla_4 X_t = Z_t + \theta Z_{t-4},$$

where $\{Z_t\} \sim WN(0, \sigma^2)$ and θ is a parameter restricted by $|\theta| < 1$. Show that $X_t(1) = E_{t-3}$ is the best one-step ahead linear predictor. [6]

(ii) Show that

$$X_t(2) = E_{t-2},\ X_t(3) = E_{t-1} \text{ and } X_t(4) = E_t,$$

gives the m-step ahead best linear predictors $X_t(m)$ for $2 \le m \le 4$. [5]

(iii) Show that $X_t(m) = X_t(m-4)$ for all $m \ge 5$. [2]

(iv) Comment on the possibility of the presence of a projected non-seasonal trend component for this model. [2]

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