# UNIVERSITY OF SURREY ©

**B. Sc. Honours Courses in Mathematical Studies**

**Level HE3 Examination**

Module MS331 BAYESIAN STATISTICS

Time allowed - 2 hours Autumn Semester 2005

Attempt THREE questions. If any candidate attempts more than THREE questions only the best THREE solutions will be taken into account.
A formula sheet and Cambridge Statistical Tables will be provided.

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**Question 1**

(a) The lifetime $T$ of a certain aircraft component follows a Weibull distribution with distribution function given by

$$F(t) = 1 - \exp(-\lambda t^{\gamma})$$

where $\lambda > 0$, $\gamma > 0$ are unknown parameters. Obtain the density function of $T$. [2]

(b) A number of $n$ components were put on test and failed at times $t_1, t_2, \ldots, t_n$. Find the likelihood function. [3]

(c) Assume that $\lambda$ and $\gamma$ are *a priori* independent. The prior density $p(\gamma)$ for $\gamma$ is not specified, but the improper prior density $p(\lambda) \propto \lambda^{-1}$ is adopted for $\lambda$. Obtain the form of the marginal posterior density for $\gamma$. (Do not attempt to evaluate the constant of proportionality.) [7]

(d) Assume now that $\gamma$ is known. Obtain the posterior distribution of $\lambda$. Suppose that the parameter of interest is the median lifetime, $\theta$. Show that

$$\theta = \left(\lambda^{-1} \log 2\right)^{1/\gamma}$$

and hence find the posterior mean of $\theta$ as a function of $\gamma$. [13]

**Question 2**

(a) A series of $n$ independent trials have outcomes $y_1, y_2, \ldots, y_n$, where $y_i = 1$ if trial $i$ results in a success, and $y_i = 0$ otherwise. Suppose that the constant chance of success in these trials is $\theta$. If the prior distribution for $\theta$ is Beta with parameters $a$ and $b$, obtain the posterior distribution of $\theta$. [9]

(b) Obtain the joint predictive distribution of the results $y_{n+1}$, $y_{n+2}$ of the next two trials in the series. Deduce that the predictive probability $\pi$ that both trials result in a failure is given by

$$\pi = \frac{(b+n-r+1)(b+n-r)}{(a+b+n+1)(a+b+n)}$$

where $r$ is the number of successes in the first $n$ trials. [9]

(c) If $r = 3$ and $n = 10$, calculate $\pi$ using

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(i) $a = b = 2$,

(ii) $a = b = 0$.

Also calculate $\pi$ in both of these cases if $r = 30$ and $n = 100$.
Comment carefully on your answers. [7]

**Question 3**

(a) Define the term sufficient statistic as applied to the family $p(\underline{x}|\theta)$. Show that if $t(\underline{x})$ is sufficient for the family $p(\underline{x}|\theta)$ then for any prior distribution, the posterior distributions of $\theta$ given $\underline{x}$ and $t(\underline{x})$ are the same. [6]

(b) Suppose $x_1, x_2, \ldots, x_n$ is a random sample from a Normal distribution with known mean $\mu$ and unknown precision $\theta$. If the prior distribution for $\theta$ is gamma, $Ga\left(\frac{a}{2}, \frac{b}{2}\right)$, find the posterior distribution of $\theta$ and deduce that the posterior distribution for $\theta$ depends on the data only though $\sum(x_i - \mu)^2$. [5]

Find the posterior mean and variance of $\theta$. [3]

Show that the posterior mean can be written as a weighted average of the prior mean and the maximum likelihood estimator $\hat{\theta}$ of $\theta$, where

$$\hat{\theta} = \frac{n}{\sum(x_i - \mu)^2}.$$

[6]

(c) If $\sum(x_i - \mu)^2 = 1$, $a = 12$, $b = 4$ and $n = 4$, find a 95% credible interval for $\theta$. [You may use the fact that if $X \sim Ga\left(\frac{a}{2}, \frac{b}{2}\right)$, then $bX \sim \chi^2_a$.] [5]

**Question 4**

(a) A three stage linear model is given by

$$\begin{aligned} \underline{y}|\,\underline{\theta}_1 &\sim N(A_1\underline{\theta}_1, C_1) \\ \underline{\theta}_1|\,\underline{\theta}_2 &\sim N(A_2\underline{\theta}_2, C_2) \\ \underline{\theta}_2 &\sim N(\underline{\mu}, C_3) \end{aligned}$$

where $\underline{y} = (y_1, y_2, \ldots, y_n)^T$, $\underline{\theta}_1$ and $\underline{\theta}_2$ are both vectors of parameters of length $p_1$ and $p_2$ respectively, $A_1$ is an $n \times p_1$ matrix, $A_2$ is a $p_1 \times p_2$ matrix, $C_1$ is an $n \times n$ variance-covariance matrix, $C_2$ is a $p_1 \times p_1$ variance-covariance matrix, $C_3$ is a $p_2 \times p_2$ variance-covariance matrix and $\underline{\mu}$ is a $p_2$ dimensional vector. $A_1, A_2, C_1, C_2, C_3$ and $\underline{\mu}$ are known.

With vague prior information i.e. $C_3^{-1} \to 0$, the posterior distribution of $\underline{\theta}_1|\underline{y}$ is $N(D_0\underline{d}_0, D_0)$ where

$$D_0^{-1} = A_1^T C_1^{-1} A_1 + C_2^{-1} - C_2^{-1} A_2 \left( A_2^T C_2^{-1} A_2 \right)^{-1} A_2^T C_2^{-1}$$

and

$$\underline{d}_0 = A_1^T C_1^{-1} y.$$

Consider a one-way model, $y_i \sim N(\theta_i, \sigma^2)$, $i = 1, 2, \ldots, n$, where $y_i$ is the mean of $m$ observations for treatment group $i$. The $\theta_i$ are iid as $N(\mu, \tau^2)$ and the prior for $\mu$ is vague. Write this model as a three stage linear model. [3]

Calculate $D_0^{-1}$ and $\underline{d}_0$ and determine $D_0$ using the result:

$$(aI_n + bJ_n)^{-1} = \frac{1}{a} I_n - \frac{b}{(a + nb)a} J_n$$

for $a > 0$, $b \neq -a/n$, where $I_n$ is the $n \times n$ identity matrix and $J_n$ is the $n \times n$ matrix whose elements all have value one. [4]

Write down the posterior mean and variance of $\theta_i$ and compare them with their classical counterparts, ie $\hat{\theta}_i = y_i$ and $Var[\hat{\theta}_i] = \sigma^2$. [6]

Write down $Cov[\theta_i, \theta_j | y]$. [1]

(b) Suppose

$$y_1 |\, \theta_1 \sim N(\theta_1, 1)$$
$$y_2 |\, \theta_2 \sim N(\theta_2, 1),$$

where $y_1$ and $y_2$ are independent, the prior for $(\theta_1, \theta_2)$ is given by

$$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \sim N\left( \begin{pmatrix} \mu \\ \mu \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \right)$$

and the prior for $\mu$ is vague. Find the joint posterior distribution of $(\theta_1, \theta_2)$ using the results in (a). [6]

Hence find the posterior distribution of $\theta_1 - \theta_2$. [5]

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**Question 5**

(a) Suppose that you wish to estimate the posterior means of three parameters, $\theta$, $\delta$ and $\beta$ in a probability model for some data. The full conditional distributions,

$$p(\theta|\delta, \beta, \text{data}), \quad p(\delta|\beta, \theta, \text{data}), \quad p(\beta|\theta, \delta, \text{data})$$

are known.

Explain carefully how to simulate a sample of the parameters from the posterior distribution using the conditional distributions to construct a Gibbs sampler. [8]

How can the posterior expectation $E[\theta^g|data]$, where $g$ is known, be estimated from this sample? [2]

(b) Suppose that we have independent observations $x_1, x_2, \ldots, x_n$ from a Poisson distribution with mean $\theta_1$ and an independent sample $y_1, y_2, \ldots, y_n$ from a Poisson distribution with mean $\theta_2$. The prior distributions of the $\theta_i$ are independent and identically distributed gamma distributions, with parameters $\alpha$ and $\beta$, where $\alpha$ is known, but $\beta$ is unknown and assigned a gamma distribution with parameters $\gamma$ and $\delta$, where $\gamma$ is known and $\delta$ is unknown. Derive the necessary conditional distributions for Gibbs sampling and explain how this would proceed. [10]

(c) Suppose that two observations $x_{n-1}, x_n$ are missing from the first sample. Explain how you would amend your algorithm to take account of the missing data. [5]

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