UNIVERSITY OF SURREY©

B. Sc. Undergraduate Programmes in Mathematical Studies
M. Math. Undergraduate Programmes in Mathematical Studies

Level HE3 Examination

Module MS315 LAGRANGIAN AND HAMILTONIAN DYNAMICS

Time allowed – 2 hrs

Autumn Semester 2007

Attempt THREE questions
If a candidate attempts more than THREE questions only the best THREE questions will be taken into account.

Question 1

(a) A particle of mass m slides under gravity on a smooth wire of shape

$$z = \cosh x,$$

where the $x-$axis is horizontal and the $z-$axis is vertically upwards.

i) Show that the Lagrangian L of the motion for the above particle is given by

$$L = \frac{m}{2}\dot{x}^2 \cosh^2 x - mg \cosh x,$$

where g is the constant gravitational acceleration and the dot denotes differentiation with respect to time. [5]

ii) Find the corresponding Hamiltonian. [3]

(b) Consider a bead of mass $m > 0$, constrained to move on a frictionless circular wire of radius a under gravitational forces. The wire is made to rotate about a vertical diameter with constant angular velocity ω. At time t the plane of the wire has turned through an angle ϕ (so that $\omega = \dot{\phi}$) and the radius to the bead makes an angle θ, $0 \leq \theta < 2\pi$, with the downward drawn vertical.

i) Show that the Lagrangian for the motion is

$$L = \frac{ma^2}{2}\left(\dot{\theta}^2 + \omega^2 \sin^2 \theta\right) + mga \cos \theta,$$

where g is the constant acceleration due to gravity, and the dots denote differentiation with respect to time. [7]

ii) By denoting by $p = \dfrac{\partial L}{\partial \dot{\theta}}$ the conjugate canonical momentum corresponding to the Lagrangian coordinate θ, find the corresponding Hamiltonian and Hamilton's equation of motion. [5]

iii) Hence find the frequency of small oscillations about the equilibrium solution $(\theta, p) = (0, 0)$. [5]

Question 2

(a) Consider the system having Lagrangian

$$L(q,\dot{q},t) = \frac{1}{2}G(q,t)\dot{q}^2 + F(q,t)\dot{q} - V(q,t).$$

Find the corresponding Hamiltonian. [4]

(b) Consider the time-independent canonical transformation from the (q,p) representation to the (Q,P) representation, with corresponding conservative Hamiltonians functions $H(q,p)$ and $K(Q,P)$ describing the same flow in each representation. Show that

$$\frac{\partial H}{\partial P} = \frac{\partial K}{\partial P} \quad \text{and} \quad \frac{\partial H}{\partial Q} = \frac{\partial K}{\partial Q},$$

so that we have $K(Q,P) = H(q(Q,P), p(Q,P))$, up to an integration constant which is irrelevant for the equations of motion. [8]

(c) Determine which of the following transformations are canonical

i) $Q = \frac{1}{2}q^2, \qquad P = \frac{p}{q};$ [1]

ii) $Q = p\tan q, \qquad P = (p-3)\cos^2 q;$ [2]

iii) $Q = \sqrt{q}e^t \cos p, \qquad P = \sqrt{q}e^{-t}\sin p$, where t is the time variable. [2]

(d) Consider a particle moving in a cartesian three-dimensional space. The components of the position vector of the particle at any time t are given by $\mathbf{r} = (x,y,z)$ and the components of its linear momentum are given by $\mathbf{p} = (p_x, p_y, p_z)$. Determine the Poisson brackets formed from the components of the angular momentum $\mathbf{J} = \mathbf{r} \times \mathbf{p}$, namely $[J_x, J_y], [J_y, J_z], [J_z, J_x]$. [8]

Question 3

a) Give the definition of symplectic matrix. [2]

b) Prove that if A and B are two symplectic matrices, then their product $C = AB$ is also a symplectic matrix. [4]

c) Consider the transformation from a fixed reference frame (q, p) to a moving reference frame (Q, P) given by $Q = q - D(t)$, where $D(t)$ is the distance between the origins of the coordinate systems at time t.

i) Show that $F_2(P, q, t) = P(q - D(t))$ is a generating function for the above transformation. [4]

ii) Given the Hamiltonian $H = \frac{p^2}{2m} + V(q)$ in the (p, q) representation, find its transformed form in the (P, Q) representation. [4]

iii) Find the equations of motion in the (P, Q) representation. [3]

d) Given a Hamiltonian system $H(\mathbf{p}, \mathbf{q}, t)$, where $\mathbf{p} = (p_1, p_2, ..., p_n)$ and $\mathbf{q} = (q_1, q_2, ..., q_n)$, prove Poisson's theorem, namely that if $f(\mathbf{p}, \mathbf{q}, t)$ and $g(\mathbf{p}, \mathbf{q}, t)$ are two integrals of motion for the above Hamiltonian, then their Poisson bracket is likewise an integral of motion. [8]

Question 4

(a) Let L and $\overline{L}$ be Lagrangians such that

$$\overline{L} - L = \frac{df(q,t)}{dt},$$

where $f(q,t)$ is a differentiable function of the generalized coordinate q and time t. Show that L and $\overline{L}$ describe the same motion. [4]

(b) Consider the Hamiltonian function $H(q,p) = \dfrac{p^2}{2q^2} + \dfrac{q^4}{8}$.

i) Find Hamilton's equations of motion. [1]

ii) By using a generating function of the first kind $F_1(q,Q) = \dfrac{q^2 Q}{2}$, transform the above Hamiltonian from the (q,p) representation into the Hamiltonian $K(Q,P)$ in the (Q,P) representation. [6]

iii) By using the new Hamiltonian $K(P,Q)$, solve the equations of motion with initial conditions $q(0) = 1,\ p(0) = 0$. [6]

iv) Solve the equations of motion found in $i)$ by using the method of Hamilton-Jacobi. [8]

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