UNIVERSITY OF SURREY ©

B. Sc. Undergraduate Programmes in Mathematical Studies

Level HE3 Examination

Module MS308 NUMERICAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

Time allowed – 2 hours

Spring Semester 2007

Attempt THREE questions.
If any candidate attempts more than THREE questions, only the best THREE solutions will be taken into account.

Question 1

Suppose that $u(x,y)$ is a solution of the PDE

$$u_t = u_{xx} - u \quad \text{for} \quad x \in (0,1) \quad \text{and} \quad t > 0,$$

satisfying the boundary conditions $u(0,t) = 1, u(1,t) = 0$ and the initial condition $u(x,0) = 1 - x$.

(a) Show that a Forward Time Central Space (FTCS) finite difference scheme for this PDE may be written

$$u_j^{n+1} = u_j^n(1 - \Delta t - 2r) + r\left(u_{j+1}^n + u_{j-1}^n\right),$$

where $r = \Delta t/\Delta x^2$. [3]

(b) Determine for what values of Δt the FTCS numerical method is stable. [9]

(c) Given a space mesh-size $\Delta x = 1/3$ and a timestep $\Delta t = 1/10$, use the FTCS method to find a numerical solution of the PDE at $t = 1/10$. [8]

(d) Make a rough sketch of the numerical solution at $t = 0$ and $t = 1/10$. What would you expect the solution to look like after many timesteps? Why? [5]

Question 2

Suppose that $u(x,t)$ is a solution of the PDE

$$u_t = u_{xx} \quad \text{for} \quad x \in (0,1) \quad \text{and} \quad t > 0,$$

satisfying the boundary conditions $u_x(0,t) = 1, u(1,t) = -1$ and the initial condition $u(x,0) = x(1-2x)$.

The Du Fort-Frankel finite difference scheme for solving this PDE using a space meshsize of Δx and timestep Δt is

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} = \frac{1}{\Delta x^2}\left(u_{j-1}^n - \left(u_j^{n-1} + u_j^{n+1}\right) + u_{j+1}^n\right).$$

(a) Show that the local truncation error (LTE) of the Du Fort-Frankel scheme is given by

$$\left(r^2 - \frac{1}{12}\right) u_{tt}\Delta x^2 + \frac{\Delta t^2}{6} u_{ttt} + O\left(\Delta t^4, \Delta x^4, r^4 \Delta x^6\right),$$

where $r = \Delta t/\Delta x^2$. [9]

(b) What conditions on Δx and Δt must be satisfied for the method to be consistent? [4]

(c) Using a 'ghost' point write down an approximation for the derivative boundary condition at $x = 0$. [2]

(d) Show that the Du Fort-Frankel scheme may be written in the form

$$\mathbf{A}\mathbf{u}^{\mathbf{n+1}} = \mathbf{B}\mathbf{u}^{\mathbf{n-1}} + \mathbf{C}\mathbf{u}^{\mathbf{n}} + \mathbf{a},$$

stating clearly what the matrices $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$, and the vectors $\mathbf{u}^{\mathbf{n}}$ and $\mathbf{a}$ are. Is the method implicit or explicit? Give a reason for your answer. [10]

Question 3

(a) Suppose that $u(x,t)$ satisfies the PDE

$$u_t + a u_x = 0, \quad a > 0.$$

(i) Show that a numerical scheme that uses a Forward Time Backward Space (FTBS) finite difference approximation may be written as

$$u_j^{n+1} = (1-p)u_j^n + p u_{j-1}^n,$$

where $p = a\Delta t/\Delta x$. [2]

(ii) Calculate the local truncation error (LTE) of the numerical scheme. [5]

(b) The FTBS method for solving the equation for $u(x,y,t)$

$$u_t + a u_x + b u_y = 0, \quad a > 0, b > 0,$$

is

$$u_{j,k}^{n+1} = (1-p-q)u_{j,k}^n + p u_{j-1,k}^n + q u_{j,k-1}^n,$$

where $p = a\Delta t/\Delta x$ and $q = b\Delta t/\Delta y$.

(i) By considering solutions to the FTBS method of the form $\xi^n \exp(ij\omega_1)\exp(ik\omega_2)$, show that the amplification factor ξ satisfies

$$|\xi|^2 = 1 + 4F(\omega_1,\omega_2),$$

where

$$F(\omega_1,\omega_2) = (p^2-p)S_1^2 + (q^2-q)S_2^2 + 2pqS_1S_2c,$$

and $S_1 = \sin(\omega_1/2)$, $S_2 = \sin(\omega_2/2)$, $C_1 = \cos(\omega_1/2)$ and $C_2 = \cos(\omega_2/2)$ and $c = \cos\left((\omega_1-\omega_2)/2\right)$. [9]

(ii) Use suitable choices of the frequencies ω_1 and ω_2 to show that the conditions $0 \le p \le 1$ and $0 \le q \le 1$ are both necessary for stability. [4]

(iii) Show by completing the square or otherwise that

$$F(\omega_1,\omega_2) = -p(1-p)\left\{\left(S_1 - \frac{qc}{(1-p)}S_2\right)^2 + \left(\frac{q(1-q)}{p(1-p)} - \frac{q^2c^2}{(1-p)^2}\right)S_2^2\right\}.$$

[2]

(iv) Hence show that the scheme is stable if p and q additionally satisfy $p+q \le 1$. [3]

Question 4

Consider the equation

$$\frac{d^2u}{dx^2} + u = -f(x), \quad \text{for} \quad x \in [0,1],$$

with $u(0) = u(1) = 0$.

(a) Write down a weak formulation of this differential equation, including a definition of the inner product and the function space V used. [6]

(b) Consider a finite element approximation, $u_h(x)$, to the solution of the weak form of the differential equation, where

$$u_h(x) = \sum_{j=1}^{N-1} U_j \phi_j(x)$$

and the functions $\phi_j(x)$ form a basis for the test space $V_h \subset V$. Using the weak form, write down the general structure of the Galerkin matrix system. [6]

(c) For $N = 2$ sketch the domain and the graph of the relevant linear hat basis function. [2]

(d) If $f(x) = 1$, calculate the required elements of the Galerkin matrix system found in (b) and hence find the finite element solution. [9]

(e) Find the analytical solution to the differential equation for $f(x) = 1$ and compare your answer with your numerical solution. [2]

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