UNIVERSITY OF SURREY©

B. Sc. Undergraduate Programmes in Mathematical Studies

Level HE3 Examination

Module MS308 NUMERICAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

Time allowed – 2 hours

Spring Semester 2006

Attempt THREE questions.
If any candidate attempts more than THREE questions, only the best THREE solutions will be taken into account.

Question 1

(a) The θ method for solving the PDE

$$u_t = u_{xx}, \qquad \text{for} \quad x \in (0,1) \quad \text{and} \quad t > 0,$$

may be written as

$$u_j^{n+1} = u_j^n + (1-\theta) r \delta_x^2 u_j^n + \theta r \delta_x^2 u_j^{n+1}.$$

(i) Show that the amplification factor for the θ method is given by

$$\xi = \frac{1 - 4(1-\theta) r \sin^2\left(\frac{\omega}{2}\right)}{1 + 4\theta r \sin^2\left(\frac{\omega}{2}\right)}$$ [7]

(ii) Hence determine the values of θ for which the method is unconditionally stable. [6]

(b) Suppose that $u(x,y,t)$ is a solution of the PDE

$$u_t = u_{xx} + u_{yy}, \qquad \text{for} \quad x \in (0,1), y \in (0,1),$$

and $t > 0$ satisfying the boundary conditions $u(x,y,t) = 0$ on $x = 0, x = 1, y = 0$ and $y = 1$ and the initial condition $u(x,y,0) = 16xy(1-x)(1-y)$.

An approximation to the PDE on a square mesh with space mesh sizes $\Delta x = \Delta y = 1/N$ and time-step Δt is given by

$$u_{j,k}^{n+1} = \left(1 + r\delta_x^2\right)\left(1 + r\delta_y^2\right) u_{j,k}^n \qquad \text{for} \quad j = 1:N-1, k = 1:N-1,$$

where $u_{j,k}^n \approx u(j\Delta x, k\Delta y, n\Delta t), r = \Delta t/\Delta x^2$. Here, δ_x^2 and δ_y^2 are the second central difference operators in the x and y directions respectively.

(i) Show that the scheme is equivalent to the two sub-steps

$$\begin{aligned} v_{j,k}^n &= u_{j,k}^n + r\delta_y^2 u_{j,k}^n \quad \text{for} \quad j = 0:N, k = 1, N-1, \\ u_{j,k}^{n+1} &= v_{j,k}^n + r\delta_x^2 v_{j,k}^n \quad \text{for} \quad j = 1:N-1, k = 1, N-1. \end{aligned}$$ [3]

(ii) Set $N = 2$ and $r = 0.25$ and use the scheme to find the approximate solution at time $t = 1/16$. [9]

Question 2

Suppose that $u(x,t)$ is a solution of the PDE

$$u_t = u_{xx} \qquad \text{for} \quad x \in (0,1) \qquad \text{and} \quad t > 0,$$

satisfying the derivative boundary conditions

$$u_x(0,t) = h(t) \qquad \text{and} \qquad u_x(1,t) = h(t),$$

where h is a known function. The FTCS approximation of the PDE using space mesh-size $\Delta x = 1/N$ and time-step Δt is

$$u_j^{n+1} = u_j^n + r\delta_x^2 u_j^n$$

for $j = 1 : N-1$, where $u_j^n \approx u(j\Delta x, n\Delta t)$, δ_x^2 is the second central difference operator and $r = \Delta t/\Delta x^2$.

(a) Show that an appropriate one-sided difference approximation for the boundary condition at $x = 1$ at time $n\Delta t$ is

$$u_N^n = u_{N-1}^n + h(n\Delta t)\Delta x$$

and derive a similar approximation for the boundary at $x = 0$. [6]

(b) In the case that $h(t) = t$ and $u(x,0) = \cos \pi x$, use the FTCS scheme along with the approximation to the boundary conditions given in (a) to obtain an approximate solution at $t = 1/18$ using $N = 3$ and one timestep. [9]

(c) Given that the approximate heat at time $n\Delta t$ is given by

$$H^n = \Delta x \sum_{j=1}^{N-1} u_j^n,$$

use the equation for the FTCS method to show that

$$H^{n+1} - H^n = r\Delta x \left(u_N^n - u_{N-1}^n - u_1^n + u_0^n\right).$$ [6]

(d) Hence show, using the approximate boundary conditions, that the approximate heat is conserved. [4]

Question 3
Suppose that $u(x,t)$ satisfies the PDE

$$u_t + au_x = 0 \qquad \text{for} \quad x \in (0,1) \quad \text{and} \quad t > 0,$$

where a is constant.
An approximation of this PDE using space mesh $\Delta x = 1/N$ and time-step Δt is

$$\frac{u_j^{n+1} - \frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right)}{\Delta t} + \frac{a\left(u_{j+1}^n - u_{j-1}^n\right)}{2\Delta x} = 0,$$

for $j = 1 : N-1$, where $u_j^n \approx u(j\Delta x, n\Delta t)$.

(a) Show that the local truncation error (LTE) for the scheme is

$$\frac{\Delta t}{2}u_{tt} - \frac{\Delta x^2}{2\Delta t}u_{xx} + \frac{a\Delta x^2}{3!}u_{xxx} + O(\Delta t^2, \Delta x^4/\Delta t, \Delta x^4).$$ [8]

(b) Show that the approximation can be written as

$$u_j^{n+1} = \frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right) - \frac{p}{2}\left(u_{j+1}^n - u_{j-1}^n\right),$$

where $p = a\Delta t/\Delta x$. [2]

(c) Show that the numerical scheme is stable if and only if $p^2 \leq 1$. [8]

(d) Show that it is possible for this numerical scheme to be stable, but not consistent with the PDE. When will the scheme converge? [7]

Question 4
Consider the differential equation

$$u''(x) = -f(x) \qquad x \in (a, b),$$

with boundary conditions

$$u(a) = u(b) = 0.$$

(a) Write down a weak formulation of this differential equation, including a definition of the inner product and the function space V used. [6]

(b) Consider a finite element approximation, $u_h(x)$, to the solution of the weak form of the differential equation, where

$$u_h(x) = \sum_{j=1}^{N-1} U_j \phi_j(x)$$

and the functions $\phi_j(x)$ form a basis for the test space $V_h \subset V$. Using the weak form, write down the general structure of the Galerkin matrix system. [6]

(c) If $a = 1, b = 2$ and $f(x) = \sin \pi x$ sketch the domain and the graphs of the relevant linear hat basis functions when $N = 2$. Calculate the required elements of the Galerkin matrix system found in (b) and hence find the finite element solution. [9]

(d) Find the analytical solution to the differential equation and compare your answer with your numerical solution. [4]

$$\left(\text{In this question you may use the fact that} \quad \int_a^b x \sin \pi x dx = \frac{1}{\pi^2} \left[\sin \pi x - x\pi \cos \pi x \right]_a^b \right)$$

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