UNIVERSITY OF SURREY©

B. Sc. Undergraduate Programmes in Mathematical Studies
M. Math. Undergraduate Programmes in Mathematical Studies

Level HE3 Examination

Module MS306 MATHEMATICAL ECONOMICS

Time allowed – 2 hours

Spring Semester 2008

Answer any **three** of the five questions.

If you attempt more than three questions, only your BEST THREE answers will be taken into account.

Each question carries 30 marks.

Approved calculators may be used.

Question 1

(a) A firm uses strictly positive quantities K and L of two inputs.

The greatest output that can be produced is $Q(K, L) = 20K^{\alpha}L^{\beta}$ where $\alpha > 0$, $\beta > 0$ and $\alpha + \beta < 1$.

(i) Show that Q is a concave function of K and L. [7]

(ii) Verify Euler's theorem for this function. [4]

The costs of the inputs are respectively £1 and £2 per unit and the value of each unit of output is £2, so the firm's profit function is

$\pi(K, L) = 40K^{\alpha}L^{\beta} - K - 2L$.

(iii) If $\alpha = \dfrac{1}{2}$ and $\beta = \dfrac{1}{4}$ find, with justification, the maximum value of $\pi(K, L)$. [8]

(b) A real-valued function f defined on a subset S of $\mathbb{R}^n$ is called **concave-contoured** if, for all $a \in \mathbb{R}$, the set $X = \{\mathbf{x} \in S \ : \ \mathrm{f}(\mathbf{x}) \geq a\}$ is a convex set.

(i) Show that if a function is concave on S then it is concave-contoured. [5]

(ii) Justify graphically that $\mathrm{f}(x, y) = xy$, defined on $\{(x, y) \in \mathbb{R}^2 \ : \ x > 0, y > 0\}$, is concave-contoured. Show that it is, however, not a concave function. [6]

Question 2

(a) Consider the following non-linear programming problem:

Maximize $z = \ln x_1 - \dfrac{2}{x_2} + x_3$ over $\mathbb{R}^3_+$, subject to the constraints

$$x_1{}^2 + x_2 \leq 3, \quad x_1{}^2 - x_2 \leq 4, \quad x_3{}^2 \leq 1.$$

(i) Use the Kuhn-Tucker necessity theorem to obtain a set of equations and inequalities which must hold at a solution of this problem. Denote the multipliers for the three constraints by $\lambda_1, \lambda_2, \lambda_3$ respectively. [5]

(ii) Show that $\lambda_3 \neq 0$ and deduce that $x_3 = 1$ at the optimal point. [3]

(iii) Show that $\lambda_1 + \lambda_2 = \dfrac{1}{2x_1{}^2}$ and $\lambda_1 - \lambda_2 = \dfrac{2}{x_2{}^2}$. Deduce that $\lambda_1 > 0$. [4]

(iv) Show that there is a solution when $\lambda_2 = 0$. Deduce, with clear justification, that the constrained maximum value of z is 0. [7]

(b) Let $\mathrm{f}(\mathbf{x}) = \mathbf{x}^t\mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^n$.

$\mathrm{f}(\mathbf{x})$ is to be maximized subject to the constraint $\mathbf{x}^t\mathbf{A}\mathbf{x} \leq c$, where $\mathbf{A}$ is a positive definite symmetric $n \times n$ real matrix and $c > 0$.

Show that the constrained maximum value of $\mathrm{f}(\mathbf{x})$ is $\dfrac{c}{\mu}$ where μ is the smallest eigenvalue of $\mathbf{A}$. [11]

Question 3

(a) The payoffs for player A in a two-player zero-sum game against player B are given by the following table:

		Player B			
		B_1	B_2	B_3	B_4
Player A	A_1	1	2	2	3
	A_2	7	1	6	0
	A_3	4	0	1	5
	A_4	8	4	k	7

Find the optimal pure or mixed strategy for player A and the value of the game in each of the cases:

(i) $k = 5$, [3]

(ii) $k = 3$ [Hint: first delete any dominated rows and columns]. [8]

(b) Let $\mathbf{A}$ be the payoff matrix for player A in a two-player zero sum game in which each of players A and B has n available pure strategies.

Let $\mathbf{u}$ denote the column vector in $\mathbb{R}^n$ whose entries are all 1.

(i) Express the problems of finding optimal mixed strategies for the two players in linear programming form. Proofs are not required, but you should give clear definitions of any extra symbols used. [5]

(ii) Show that if the equations $\mathbf{A}^t\mathbf{x} = \mathbf{u}$ and $\mathbf{A}\mathbf{y} = \mathbf{u}$ have non-negative solutions $\mathbf{x}^*$ and $\mathbf{y}^*$ then these solutions determine the optimal mixed strategies. [6]

(iii) If $\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & -2 \\ 0 & 0 & 3 \end{pmatrix}$, solve the equations $\mathbf{A}^t\mathbf{x} = \mathbf{u}$ and $\mathbf{A}\mathbf{y} = \mathbf{u}$.

Hence, using the result in (ii), find the value of the game and the optimal mixed strategy for each player. [8]

Question 4

(a) Let $\mathbf{p}$ and $\mathbf{q}$ be mixed strategies for the players in a two-player general-sum game in which the $m \times n$ payoff matrices for the players are $\mathbf{A}$ and $\mathbf{B}$ respectively.

State the properties that $(\mathbf{p}, \mathbf{q})$ must have to be an *equilibrium pair*, and briefly explain what this means in terms of the players' choices of strategy. [4]

(b) Let $C_i = \max(0, \mathbf{e}_i{}^t\mathbf{A}\mathbf{q} - \mathbf{p}^t\mathbf{A}\mathbf{q})$ for $i = 1, \ldots, m$ and $D_j = \max(0, \mathbf{p}^t\mathbf{B}\mathbf{e}_j - \mathbf{p}^t\mathbf{B}\mathbf{q})$ for $j = 1, \ldots, n$, where $\mathbf{e}_i$ has ith entry 1 and all other entries 0.

Define a pair $(\mathbf{p}', \mathbf{q}')$ of mixed strategies such that $(\mathbf{p}, \mathbf{q})$ is an equilibrium pair if and only if $(\mathbf{p}', \mathbf{q}') = (\mathbf{p}, \mathbf{q})$. [3]

(c) State the Brouwer fixed-point theorem and describe, in outline, how it is used to show that an equilibrium pair exists in every two-player general-sum game. [8]

(d) Now take $m = 2$ and suppose $(\mathbf{p}, \mathbf{q})$ is an equilibrium pair, where $\mathbf{p} = \begin{pmatrix} p \\ 1-p \end{pmatrix}$ and $0 < p < 1$. If $\mathbf{A}\mathbf{q} = \begin{pmatrix} x \\ y \end{pmatrix}$, show that $x = y$. [6]

(e) Consider the two-player general-sum game represented by the following bimatrix, where an ordered pair (α, β) denotes payoffs of α to Player A and β to Player B.

		Player B		
		B_1	B_2	B_3
Player A	A_1	$(1,0)$	$(1,2)$	$(2,0)$
	A_2	$(0,1)$	$(2,0)$	$(0,1)$

Suppose $(\mathbf{p}, \mathbf{q})$ is an equilibrium pair where $\mathbf{p} = \begin{pmatrix} p \\ 1-p \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \\ 1-q_1-q_2 \end{pmatrix}$.

Show that $\frac{1}{2} \leq q_2 \leq \frac{2}{3}$ and find $\mathbf{p}$ when $q_2 = \frac{1}{2}$. [9]

Question 5

(a) The technology matrix for a three-sector open economy is

$$\mathbf{A} = \begin{pmatrix} 0.2 & 0.3 & 0.5 \\ 0.1 & 0 & 0 \\ 0 & 0 & 0.3 \end{pmatrix},$$

i.e. each unit of output in Sector 1 requires inputs of 0.2 unit from Sector 1 and 0.1 from Sector 2, etc.

(i) Find the eigenvalues of $\mathbf{A}$, showing that they all have modulus less than 1. State, with justification, what this tells you about the economy. [6]

(ii) There is an exogenous demand for 500 units from Sector 1, 400 units from Sector 2 and 700 units from Sector 3. Find, to the nearest unit, the output that each sector must produce. [8]

(b) Let $\mathbf{C}$ be a stochastic $n \times n$ real matrix and suppose $\mathbf{x} \in \mathbb{R}^n$ is an eigenvector of $\mathbf{C}^t$ associated with a real eigenvalue λ. Let x_k be the largest entry of $\mathbf{x}$.

(i) By considering the kth entry of $\mathbf{C}^t\mathbf{x}$, show that $\lambda \leq 1$. [7]

(ii) Hence show that the largest real eigenvalue of a real stochastic matrix is 1. [4]

(iii) It is known that $\mathbf{C}$ has a non-negative eigenvector associated with the eigenvalue 1. Explain the significance of this for the closed Leontief model. [2]

(iv) Suppose the technology matrix for a closed economy is doubly stochastic, i.e. its row sums and column sums are all 1.

Show that the economy is in equilibrium when the prices of the outputs are all equal. [3]

INTERNAL EXAMINER: Dr D.J. Fisher
EXTERNAL EXAMINER: Dr D. Chillingworth