UNIVERSITY OF SURREY©

B. Sc. Undergraduate Programmes in Mathematical Studies

Level HE3 Examination

MS306 Mathematical Economics

Time allowed - 2 hours | Autumn Semester 2006

Answer any **three** of the five questions

If you attempt more than three questions, only your BEST THREE answers will be taken into account.

Each question carries 30 marks.

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Question 1

(a) For a production function $Q(x_1, \ldots, x_n)$ defined on $\mathbb{R}^n_+$, give definitions of

(i) the **production possibility set**, [2]

(ii) an **isoquant**, [2]

(iii) the **marginal product** of x_i. [2]

Now let $Q(x_1, \ldots, x_n) = \ln x_1 + \ln x_2 + \cdots + \ln x_n$.

(iv) Determine, with explanation, whether or not Q is a homogeneous function. [3]

(v) Determine, with explanation, whether or not Q is a concave function. [3]

(vi) $Q(x_1, \ldots, x_n)$ is to be maximized subject to the constraint $\sum_{i=1}^{n} x_i = c$ where c is a constant.
Show that the constrained maximum value is $n \ln \frac{c}{n}$ [9]

(b) The function $\mathrm{f} : \mathbb{R}^n \to \mathbb{R}$ is given by $\mathrm{f}(\mathbf{x}) = \mathbf{x}^t(\mathbf{A} + \mathbf{I})(\mathbf{x} + \mathbf{b})$, where $\mathbf{A}$ is a positive definite symmetric $n \times n$ real matrix, $\mathbf{I}$ is the $n \times n$ identity matrix and $\mathbf{b}$ is a constant vector in $\mathbb{R}^n$.

(i) Find, in terms of $\mathbf{b}$, the vector $\mathbf{x}^*$ such that $\nabla \mathrm{f}(\mathbf{x}^*) = \mathbf{0}$. [7]

(ii) Determine, with explanation, whether $\mathbf{x}^*$ is a global minimizer of $\mathrm{f}(\mathbf{x})$ over $\mathbb{R}^n$. [2]

Question 2

(a) The function f is defined on $\mathbb{R}^2_+$ by $\mathrm{f}(x,y) = xy + x^2 - x^3 - y^3$.
S is the largest subset of $\mathbb{R}^2_+$ on which f is a concave function.

(i) Find two inequalities which define S. [6]

(ii) If $x^* = \dfrac{2}{3}$ and $y^* = \dfrac{1}{3}$, show that (x^*, y^*) maximizes $\mathrm{f}(x,y)$ over S subject to the constraints $x + y \leq 1$ and $2x + y \leq 2$. [9]

(b) (i) The quadratic form $\mathbf{x}^t\mathbf{Q}\mathbf{x}$, where $\mathbf{Q}$ is a positive definite symmetric $n \times n$ matrix, is to be minimized subject to the constraints $\mathbf{Ax} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$, where $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{b} \in \mathbb{R}^m$.

Formulate the Kuhn-Tucker conditions for this problem. Hence show that the problem can be solved by finding non-negative vectors $\mathbf{x}, \boldsymbol{\mu}, \mathbf{y}, \boldsymbol{\lambda}$ such that $\boldsymbol{\mu}^t\mathbf{x} = \boldsymbol{\lambda}^t\mathbf{y} = 0$ and

$$\begin{pmatrix} -2\mathbf{Q} & \mathbf{I}_3 & \mathbf{O} & -\mathbf{A}^t \\ \mathbf{A} & \mathbf{O} & \mathbf{I}_2 & \mathbf{O} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\mu} \\ \mathbf{y} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{b} \end{pmatrix}.$$ [9]

(ii) A portfolio is created by investing £x_i in shares of type S_i for $i = 1, 2, 3$.

X_1, X_2, X_3 are random variables representing the annual return, in pounds, on £1 invested in S_1, S_2, S_3 respectively.

The mean values of X_1, X_2, X_3 are $0.10, 0.08, 0.07$ respectively.

The variances of X_1, X_2, X_3 are $0.09, 0.05, 0.12$ respectively.

The covariances are $\sigma_{12} = -0.01, \sigma_{13} = 0.03, \sigma_{23} = 0.02$.

At most £1000 is to be invested. An overall return of at least £100 is required after a year, at minimum risk.

Find $\mathbf{Q}$, $\mathbf{A}$ and $\mathbf{b}$ such that this problem has the form:

Minimize $\mathbf{x}^t\mathbf{Q}\mathbf{x}$ subject to $\mathbf{Ax} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$. [6]

[Do NOT attempt to solve the problem.]

Question 3

(a) The payoff matrix for a two-player zero-sum game is:

		Player B		
		B_1	B_2	B_3
Player A	A_1	$c+2$	1	2
	A_2	2	c	-2

(i) When $c = -3$, show that there is a dominated strategy. Find the optimal mixed strategy for each player in this case, and the value of the game. [7]

(ii) Find the range of values of c for which the game has a solution in pure strategies. State the value of the game in this case. [6]

(b) Each of two players A and B independently chooses one of the numbers 1, 2, 3. If the numbers are equal, B pays A an amount equal to that number. If they are not equal, A pays B an amount equal to B's number. It is known that the value of this game to A is $-\frac{6}{11}$.

(i) Construct the payoff matrix for A, modify it if necessary, and hence express the problem of finding A's optimal mixed strategy in Linear Programming form. [7]

(ii) Given that A's optimal mixed strategy is to choose 1, 2 and 3 with probabilities $\frac{5}{22}, \frac{4}{11}, \frac{9}{22}$ respectively, find B's optimal mixed strategy. [10]

Question 4

(a) Every Friday evening, Arnold and Basil go to one of two pubs: the Crown (which they both prefer) or the Dolphin. Arnold likes to tell jokes to Basil, but Basil finds Arnold's jokes boring (though he would never say so).

Thus Arnold's preferences are to be (1) at the Crown with Basil, (2) at the Dolphin with Basil, (3) at the Crown without Basil, (4) at the Dolphin without Basil.

Basil's preferences are to be (1) at the Crown without Arnold, (2) at the Dolphin without Arnold, (3) at the Crown with Arnold, (4) at the Dolphin with Arnold.

Assigning a payoff of 4 for the first preference, 3 for the second, 2 for the third and 1 for the fourth, and assuming that each 'player' decides where to go without consulting the other,

(i) express this situation as a general-sum game in bimatrix form. Also express it in extensive form using a Kuhn tree, indicating any information sets. [7]

(ii) Determine, with explanation, whether there are any dominant-strategy equilibria or Nash equilibria. [4]

(iii) Find and interpret the safety level for each 'player'. [5]

(b) In the following payoff matrix for a two-player general-sum game, an entry of the form (x, y) means that player A receives a payoff x and player B receives a payoff y.

		Player B	
		B_1	B_2
	A_1	$(3, 2)$	$(-2, 3)$
Player A	A_2	$(2, -1)$	$(3, -2)$
	A_3	$(-4, 3)$	$(4, -3)$

(i) Explain what is meant by an **equilibrium pair** for this game. [4]

(ii) Given that this game has an equilibrium pair in which Players A and B use the mixed strategies $(p_1, p_2, 1 - p_1 - p_2)$ and $\left(\frac{1}{2}, \frac{1}{2}\right)$ respectively, show that $7p_1 + 5p_2 = 6$ and find the range of possible values of p_1. [10]

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Question 5

(a) Let $\mathbf{C}$ be a stochastic $n \times n$ matrix and let X be the set of non-negative vectors in $\mathbb{R}^n$ whose entries have sum 1.

(i) Show that if $\mathbf{x} \in X$ then $\mathbf{Cx} \in X$. [5]

(ii) Use the Brouwer fixed-point theorem to deduce that 1 is an eigenvalue of $\mathbf{C}$ associated with a non-negative eigenvector. [4]

(iii) State the significance of this result for the closed Leontief model. [1]

(b) A closed economy with three sectors has technology matrix

$$\mathbf{A} = \begin{pmatrix} 0.2 & 0.4 & 0.4 \\ 0.4 & 0.2 & 0.4 \\ 0.2 & 0.8 & 0 \end{pmatrix}.$$

(i) State the significance of the entry 0.8 in $\mathbf{A}$. [1]

(ii) Find equilibrium prices which make each sector's output equal in value to its inputs. [7]

(iii) State, with a reason, whether $\mathbf{A}$ could be the technology matrix for an open economy. [2]

(c) An open economy with two sectors has technology matrix

$$\mathbf{B} = \begin{pmatrix} 0.8 & 0.3 \\ 0.1 & 0.6 \end{pmatrix}.$$

(i) Find the eigenvalues of $\mathbf{B}$. Hence, using a result established in the course, show that any set of exogenous demands can be met. [4]

(ii) If the economy grows at a uniform rate $r\%$, so the output vector $\mathbf{x}$ satisfies $\mathbf{x} = \left(1 + \frac{r}{100}\right)\mathbf{Bx}$, find the value of r and the corresponding vector $\mathbf{x}$. [6]

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