UNIVERSITY OF SURREY©

B. Sc. Undergraduate Programmes in Mathematical Studies
M. Math. Undergraduate Programmes in Mathematical Studies

Level HE3 Examination

Module MS304 CHAOS AND FRACTALS

Time allowed – 2 hrs Spring Semester 2006

Attempt FOUR questions
If a candidate attempts more than FOUR questions only the best FOUR questions will be taken into account.

Question 1

(a) For one-dimensional iterated maps, there are three properties that characterise chaos, one of which is the mixing property.

(i) State the other two properties. [2]

(ii) Give the definition of the mixing property. [2]

(iii) The doubling map $D : [0, 1) \to [0, 1)$ is defined by

$$D(x) = \mathrm{Frac}(2x),$$

where $\mathrm{Frac}(y)$ is the fractional part of y. The doubling map has a stronger property than mixing. State and prove this property. [7]

(iv) Does this stronger property also hold for the quadratic map $Q(x) = 4x(1 - x)$? Justify your answer. [4]

(v) Why is the concept of a topological conjugacy between two maps a useful one? [2]

(b) Consider the maps $T : [0, 1] \to [0, 1]$ and $g : [-1, 1] \to [-1, 1]$ defined by

$$T(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1/2 \\ 2 - 2x & \text{if } 1/2 < x \le 1 \end{cases} \qquad g(x) = 1 - 2x^2.$$

Show that

$$hT = gh,$$

where $h : [0, 1] \to [-1, 1]$ is defined by $h(x) = -\cos \pi x$. [6]

What is the relationship between T and g? [2]

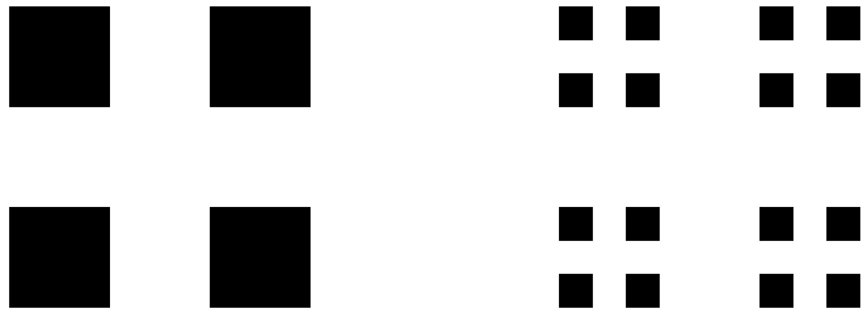

Figure 1: Construction of a fractal.

Question 2

(a) A fractal is constructed by starting with a square of side 1 and removing a cross from the middle of the square, the arms of the cross having width 1/3. A similar cross is then removed from the remaining squares and this procedure is repeated an infinite number of times. The first two steps of this construction are shown in Fig. 1. Determine the box-counting dimension of this fractal. [4]

(b) Consider the iteration $x_{n+1} = T_3(x_n)$, where $T_3 : \mathbf{R} \to \mathbf{R}$ is defined by

$$T_3(x) = \begin{cases} 3x, & x \le 1/2, \\ 3 - 3x, & x \ge 1/2. \end{cases}$$

(i) Sketch the graph of $T_3(x)$. [2]

(ii) Show that if $x_k \notin [0, 1]$ for some $k \ge 0$ then $x_n \to -\infty$ as $n \to \infty$. (Hint: Consider the two cases $x_k < 0$ and $x_k > 1$ separately.) [4]

(iii) By considering only $T_3(x)$, determine a range of initial conditions in the interval $[0, 1]$ whose orbits will diverge to $-\infty$. [3]

(iv) Find $T_3(T_3(x))$ and sketch its graph. (Hint: Consider the intervals $(-\infty, 1/6]$, $[1/6, 1/2]$, $[1/2, 5/6]$ and $[5/6, \infty)$.) [6]

(v) By considering only $T_3(T_3(x))$ determine two further intervals of initial conditions in the interval $[0, 1]$ whose orbits will diverge to $-\infty$. [4]

(vi) If this process were repeated infinitely many times, what set of initial conditions would remain whose orbits cannot be shown to diverge to $-\infty$? [2]

Question 3

(a) Describe the period-doubling route to chaos which occurs in the logistic map including a discussion of the Feigenbaum number. Why is this number important? [5]

(b) Consider the iteration

$$x_{n+1} = \frac{\lambda x_n}{x_n^2 - x_n + 1}.$$

(i) Determine the path of nonzero fixed points of this map. [2]

(ii) Find a turning point $(\tilde{x}, \tilde{\lambda})$ on this path and determine which side of the turning point the two paths of fixed points lie on. [3]

(iii) If $x_0 = \tilde{x}$, describe the behaviour of the orbit with this initial condition for (i) $\lambda < \tilde{\lambda}$ and (ii) $\lambda > \tilde{\lambda}$, assuming that the map has no other stable periodic orbits. [4]

(iv) Show that the point $(x_0, \lambda_0) = (2, 3)$ is a period-doubling bifurcation point on the path of fixed points. [3]

(c) (i) State Sarkovskii's Theorem. [3]

(ii) As a part of the proof of Sarkovskii's Theorem, we want to show that if a map g has a period k point for some $k > 1$ then it also has a period 1 point (a fixed point). Prove this result by considering the function $g(x) - x$ evaluated at some of the period k points. [5]

Question 4

(a) (i) For the scalar iterated map $x_{n+1} = g(x_n)$, state a formula for the Lyapunov exponent. [2]

(ii) What information does the Lyapunov exponent give about the dynamics of the map g? [2]

(iii) What is the significance of a positive Lyapunov exponent? [2]

(iv) Calculate the Lyapunov exponent when $g(x) = D(x) := \mathrm{Frac}(2x)$. [4]

(b) Consider the two maps

$$x_{n+1} = f(x_n), \qquad f : I_1 \to I_1,$$

$$y_{n+1} = g(y_n), \qquad g : I_2 \to I_2,$$

where f and g are assumed to be differentiable.

(i) If $h : I_1 \to I_2$, state the conditions required for g to be topologically semi-conjugate to f via h. [2]

(ii) Write down the formula for the Lyapunov exponent σ_f of f along the orbit $x_0, x_1, \ldots, x_k, \ldots$. [2]

(iii) If g is topologically semi-conjugate to f via h and h is also differentiable, show that

$$h'(x_{k+1})f'(x_k) = g'(h(x_k))h'(x_k).$$

[2]

(iv) Hence find $\displaystyle\prod_{k=0}^{n-1} |f'(x_k)|$ in terms of the functions g and h, simplifying your answer as far as possible. [4]

(v) Assuming that $h'(x_k) \neq 0$ for all $k \geq 0$ and that $\dfrac{1}{k} \log |h'(x_k)| \to 0$ as $k \to \infty$, show that the Lyapunov exponents for f along $x_0, x_1, \ldots, x_k, \ldots$ and for g along $h(x_0), h(x_1), \ldots, h(x_k), \ldots$ are the same. [5]

SEE NEXT PAGE

Question 5

Consider the two coupled maps

$$
\begin{aligned}
x_{n+1} &= 4x_n(1-x_n) + c(x_n - y_n)(x_n + y_n - 1) \\
y_{n+1} &= 4y_n(1-y_n) + c(y_n - x_n)(y_n + x_n - 1)
\end{aligned}
$$

(a) Show that these equations have an invariant subspace.

What is the significance of this subspace? [4]

(b) Rewrite these equations in terms of the new variables

$$X = \frac{1}{2}(x+y), \qquad Y = \frac{1}{2}(x-y).$$

How is the invariant subspace defined in terms of the new variables? [5]

(c) Find the Jacobian matrix for the equations in these new variables and evaluate the derivatives in the invariant subspace. [4]

(d) Derive the two Lyapunov exponents for this system. (Use may use the result that the Lyapunov exponent for the quadratic map $Q(x) = 4x(1-x)$ is $\log 2$.) [8]

(e) State a condition for the invariant subspace to be attracting and find two values of c at which the invariant subspace changes from being attracting to being repelling. [4]

Question 6

(a) The function Frac(x) gives the fractional part of x. Show that for any integer m,

$$\text{Frac}(m\text{Frac}(x) + c) = \text{Frac}(mx + c) \qquad (1)$$ [3]

(b) Prove by induction that the iterates of the doubling map $D(x) = \text{Frac}(2x)$, with an initial value $x_0 \in [0, 1)$, are given by

$$x_n = \text{Frac}(2^n x_0).$$ [4]

(c) The iteration $x_{n+1} = D(x_n)$, where $D(x) = \text{Frac}(2x)$, is performed on a computer and rounding errors are introduced at each stage. The approximate initial condition is given by

$$y_0 = \text{Frac}(x_0 + \epsilon_0)$$

and the approximate orbit is found from the iteration

$$y_{n+1} = \text{Frac}(2y_n + \epsilon_{n+1}),$$

where ϵ_{n+1} is the rounding error at each iteration which satisfies

$$|\epsilon_n| < \epsilon, \qquad n = 0, 1, 2, \ldots$$

for some $\epsilon > 0$.

Use proof by induction to show that

$$y_n = \text{Frac}\left(2^n x_0 + \sum_{k=0}^{n} 2^{n-k}\epsilon_k\right).$$

(Hint: You may find equation (1) useful.) [8]

(d) If z_0 is the starting value for an iteration of the doubling map D, then $z_n = \text{Frac}(2^n z_0)$. Show that z_0 can be chosen such that $z_m = y_m$ for some m.

It can also be shown that $|z_n - y_n| < \epsilon$ for all $n = 1, 2, \ldots, m$. What is the significance of this result? [6]

(e) The quadratic map $Q(x) = 4x(1 - x)$ is topologically semi-conjugate to the doubling map $D(x)$. What do the results in parts (b) and (c) combined with this relationship between Q and D tell us about the quadratic map Q? [4]

INTERNAL EXAMINER: P.J. ASTON
EXTERNAL EXAMINER: P. GLENDINNING