UNIVERSITY OF SURREY©

B. Sc. Undergraduate Programmes in Mathematical Studies

Level HE3 Examination

Module MS303 CURVES AND SURFACES

Time allowed – 2 hrs Autumn Semester 2007

Attempt THREE questions
If a candidate attempts more than THREE questions only the best THREE questions will be taken into account.

Question 1

Let $\boldsymbol{\gamma}(s)$ be a *unit-speed* space curve and suppose that the curvature $\kappa(s) > 0$ for all s. Let

$$\mathbf{t}(s) = \dot{\boldsymbol{\gamma}}(s)\,, \quad \mathbf{n}(s) = \frac{\ddot{\boldsymbol{\gamma}}(s)}{\|\ddot{\boldsymbol{\gamma}}(s)\|}\,, \quad \mathbf{b}(s) = \mathbf{t}(s) \times \mathbf{n}(s)\,.$$

The Frenet-Serret equations satisfied by the moving frame $(\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s))$ are

$$\dot{\mathbf{t}} = \kappa \mathbf{n}\,, \quad \dot{\mathbf{n}} = -\kappa \mathbf{t} + \tau \mathbf{b}\,, \quad \dot{\mathbf{b}} = -\tau \mathbf{n}\,,$$

where $\kappa(s)$ and $\tau(s)$ are the curvature and torsion respectively.

(a) Express the *third derivative* of $\boldsymbol{\gamma}(s)$ in terms of the moving frame

$$\dddot{\boldsymbol{\gamma}}(s) = a_1(s)\mathbf{t}(s) + a_2(s)\mathbf{n}(s) + a_3(s)\mathbf{b}(s)\,.$$

Find expressions for the scalar-valued functions $a_1(s), a_2(s), a_3(s)$ in terms of the curvature $\kappa(s)$, torsion $\tau(s)$ and $\dot{\kappa}$. [8]

(b) Using the expression for $\dddot{\boldsymbol{\gamma}}$ derived in (b), show that

$$\dot{\boldsymbol{\gamma}} \cdot (\ddot{\boldsymbol{\gamma}} \times \dddot{\boldsymbol{\gamma}}) = \tau\,\kappa^2\,.$$

[5]

(c) Show that

$$\dot{\boldsymbol{\gamma}} \cdot (\ddot{\boldsymbol{\gamma}} \times \dddot{\boldsymbol{\gamma}}) = -\kappa\,\dot{\mathbf{t}} \cdot \dot{\mathbf{b}}\,.$$

[4]

(d) Suppose that there exists a non-zero constant vector $\mathbf{m} \in \mathbb{R}^3$ such that $\mathbf{m} \cdot \dot{\boldsymbol{\gamma}}(s) = 0$ for all s. Show that the torsion $\tau(s)$ of the curve is zero for all s. [8]

Question 2

Let M be a surface with regular coordinate chart $(\mathbf{x}, U)$, $U \subset \mathbb{R}^2$. Take the unit normal at each point to be

$$\mathbf{n}(u,v) = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}, \quad (u,v) \in U .$$

Assume that $\mathbf{x}_u \cdot \mathbf{x}_v = 0$ at all points on the surface.

(a) Expressing the derivative of $\mathbf{n}(u,v)$ as

$$\begin{aligned} \mathbf{n}_u &= -a_{11}\mathbf{x}_u - a_{12}\mathbf{x}_v - a_{13}\mathbf{n} \\ \mathbf{n}_v &= -a_{21}\mathbf{x}_u - a_{22}\mathbf{x}_v - a_{23}\mathbf{n} . \end{aligned}$$

Explain why $a_{13} = a_{23} = 0$ and find expressions for $a_{11}, a_{12}, a_{21}, a_{22}$ in terms of

$$E = \mathbf{x}_u \cdot \mathbf{x}_u\,, \quad G = \mathbf{x}_v \cdot \mathbf{x}_v\,, \quad L = \mathbf{n} \cdot \mathbf{x}_{uu}\,, \quad M = \mathbf{n} \cdot \mathbf{x}_{uv}\,, \quad N = \mathbf{n} \cdot \mathbf{x}_{vv}\,.$$

[8]

(b) Let $\boldsymbol{\gamma}(t) = \mathbf{x}(u(t), v(t))$ be a regular curve in the surface passing through a point $\mathbf{p}$. Show that second derivative of the curve at the point $\mathbf{p}$ satisfies

$$\mathbf{n} \cdot \ddot{\boldsymbol{\gamma}} = \Pi_p(\dot{\mathbf{w}}, \dot{\mathbf{w}})\,, \quad \mathbf{w}(t) := \begin{pmatrix} u(t) \\ v(t) \end{pmatrix},$$

where $\Pi_p(\dot{\mathbf{w}}, \dot{\mathbf{w}}) = \dot{u}^2 L + 2\dot{u}\dot{v}M + \dot{v}^2 N$.

[6]

(c) How are the eigenvalues of the matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ related to curvature?

[2]

(d) Define what it means for a point $\mathbf{p} \in M$ to be *umbilic*, and relate this property to the eigenvalues of $\mathbf{A}$.

[2]

(e) Suppose $\mathbf{p}$ is an umbilic point, and let $\boldsymbol{\gamma}(t) = \mathbf{x}(u(t), v(t))$ be a curve passing through $\mathbf{p}$. Show that the normal curvature at $\mathbf{p}$ can be expressed in the form

$$\kappa_n = \frac{1}{\ell^2}\mathbf{n} \cdot \ddot{\boldsymbol{\gamma}}\,, \quad \text{with} \quad \ell = \sqrt{E\dot{u}^2 + G\dot{v}^2}\,.$$

[7]

Question 3

Let M be a *ruled surface* in $\mathbb{R}^3$. Ruled surfaces can be expressed in terms of a single coordinate chart $(\mathbf{x}, U)$ with

$$\mathbf{x}(u,v) = \boldsymbol{\gamma}(u) + v\,\boldsymbol{\delta}(u)\,,$$

where $\boldsymbol{\gamma}(u)$ and $\boldsymbol{\delta}(u)$ are given space curves, and

$$U = \{\, (u,v) \in \mathbb{R}^2 \ :\ 0 < u < \infty\,, \quad v \in \mathbb{R}\,\}\,.$$

(a) State conditions on $\boldsymbol{\gamma}(u)$ and $\boldsymbol{\delta}(u)$ for the coordinate chart to be *regular*. [5]

(b) State a formula for the Gaussian curvature in terms of E, F, G and L, M, N where

$$E = \mathbf{x}_u \cdot \mathbf{x}_u\,, \quad F = \mathbf{x}_u \cdot \mathbf{x}_v\,, \quad G = \mathbf{x}_v \cdot \mathbf{x}_v\,, \quad L = \mathbf{n} \cdot \mathbf{x}_{uu}\,, \quad M = \mathbf{n} \cdot \mathbf{x}_{uv}\,, \quad N = \mathbf{n} \cdot \mathbf{x}_{vv}\,,$$

where $\mathbf{n}$ is the unit normal vector. [4]

(c) Suppose that $\boldsymbol{\gamma}(u)$ is a unit speed curve, $\boldsymbol{\delta}(u)$ has unit length,

$$\dot{\boldsymbol{\gamma}}(t) \cdot \boldsymbol{\delta}(t) = 0 \quad \text{and} \quad \dot{\boldsymbol{\gamma}}(t) \cdot \dot{\boldsymbol{\delta}}(t) = 0\,.$$

Determine expressions for the coefficients E, F and G. [8]

(d) Prove that the Gaussian curvature of a ruled surface with a regular coordinate chart is less than or equal to zero at every point on the surface. [8]

Question 4

Consider a torus M in $\mathbb{R}^3$ with regular coordinate chart $(\mathbf{x}, U)$ of the form

$$\mathbf{x}(u,v) = ((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u)\,, \quad R > r > 0\,,$$

$U = \{(u,v) \;:\; 0 < u < 2\pi\,,\; 0 < v < 2\pi\}$. Consider a curve in M of the form

$$\boldsymbol{\gamma}(t) = \mathbf{x}(u(t), v(t))\,, \quad t \in I \subset \mathbb{R}\,,$$

with $(u(t), v(t))$ a regular parametrised curve in U.

(a) A basis for the tangent space $T_\gamma M$ at each point along the curve is given by $\{\mathbf{x}_u, \mathbf{x}_v\}$ evaluated on the curve. Give an expression for this basis on the above chart for the torus. Find an *orthonormal* basis $\{\boldsymbol{\xi}_1(t), \boldsymbol{\xi}_2(t)\}$ for the tangent space. [4]

(b) Let

$$\dot{\boldsymbol{\xi}}_1 = c_1(t)\boldsymbol{\xi}_1(t) + c_2(t)\boldsymbol{\xi}_2(t) + \nu_1(t)\mathbf{n}(t) \quad \text{and} \quad \dot{\boldsymbol{\xi}}_2 = \omega(t)\boldsymbol{\xi}_1(t) + c_3(t)\boldsymbol{\xi}_2(t) + \nu_2(t)\mathbf{n}(t)\,,$$

where $\mathbf{n}(t)$ is the normal vector at each point along the curve. Show that $c_1 = c_3 = 0$ and $c_2 = -\omega(t)$ and find an explicit expression for $\omega(t)$ for a path on the torus. It is not necessary to find expressions for $\nu_1(t)$ or $\nu_2(t)$. [7]

(c) A *vectorfield* $\mathbf{w}(t)$ along the curve in M can be expressed in the form

$$\mathbf{w}(t) = w_1(t)\boldsymbol{\xi}_1(t) + w_2(t)\boldsymbol{\xi}_2(t)\,,$$

where $w_1(t)$ and $w_2(t)$ are real-valued functions. Give conditions for the vectorfield $\mathbf{w}(t)$ to be *parallel*. [3]

(d) Find the differential equations that $w_1(t)$ and $w_2(t)$ must satisfy for a vectorfield $\mathbf{w}(t)$ to be parallel on M. [5]

(e) Suppose $0 < t < \pi$ and take $w_1(0) = 1$ and $w_2(0) = 0$. Find the coefficients $(w_1(t), w_2(t))$ for a parallel vectorfield on M along the path $u(t) = u_0$ with $u_0 \in (0, \pi)$ and $v(t) = t$. [6]

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