UNIVERSITY OF SURREY©

B. Sc. Undergraduate Programmes in Mathematical Studies

Level HE3 Examination

Module MS303 CURVES AND SURFACES

Time allowed – 2 hours Autumn Semester 2006

Attempt THREE questions
If a candidate attempts more than THREE questions only the best THREE questions will be taken into account.

Question 1

Let $\boldsymbol{\gamma}(t)$ be a space curve with $\kappa(t) > 0$ for all t. A space curve $\boldsymbol{\gamma}(t)$ is called a *general helix* if there exists a constant unit vector $\mathbf{m} \in \mathbb{R}^3$ such that $\mathbf{m} \cdot \mathbf{T}(t) = \cos\varphi$, where $\mathbf{T}(t) = \dot{\boldsymbol{\gamma}}(t)/\|\dot{\boldsymbol{\gamma}}(t)\|$, and φ is a constant angle satisfying $0 < \varphi < \pi$. The vector $\mathbf{m}$ is called the *axis* of the helix.

(a) Suppose $\gamma(s)$ is a unit speed general helix. Show that

$$\mathbf{m} = \alpha_1 \mathbf{T}(s) + \alpha_2 \mathbf{B}(s)$$

with α_1 and α_2 independent of s, where $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ is a Frenet-Serret frame for the curve. Express α_1 and α_2 as functions of φ. [5]

(b) Suppose that $\boldsymbol{\gamma}(s)$ is a unit speed general helix. Show that the curvature and torsion satisfy

$$\frac{\tau(s)}{\kappa(s)} = \mathsf{constant} \quad \forall\ s\,,$$

and determine an expression for the constant in terms of φ. [7]

(c) Let $\boldsymbol{\gamma}(s)$ be a unit speed curve with its curvature and torsion satisfying $\tau(s) = c\,\kappa(s)$ for some positive constant c. Prove that it is a general helix: show that there exists a unit vector $\mathbf{m} \in \mathbb{R}^3$ with

$$\mathbf{m} \cdot \dot{\boldsymbol{\gamma}}(s) = f(c)\,,$$

where $f(c)$ is some function of c only. Determine the function $f(c)$ and show that it satisfies $0 < f(c) < 1$. [8]

(d) Let $\boldsymbol{\gamma}(t)$ be a space curve of the form

$$\boldsymbol{\gamma}(t) = \boldsymbol{\beta}(t) + a(t)\mathbf{m}\,,$$

where $\boldsymbol{\beta}(t)$ is a space curve satisfying $\mathbf{m} \cdot \dot{\boldsymbol{\beta}}(t) = 0$ and $\|\dot{\boldsymbol{\beta}}(t)\| = 1$ for all t, $\mathbf{m}$ is a constant unit vector, and $a(t)$ is a scalar-valued function. Determine the most general form for $a(t)$ so that $\boldsymbol{\gamma}(t)$ is a general helix, with axis $\mathbf{m}$. [5]

Question 2
Consider the surface $\mathcal{S}$ in $\mathbb{R}^3$ defined by the graph of the function

$$h(u,v) = u^2 + uv + \frac{1}{2}v^2 - \frac{1}{12}u^3 \,.$$

This surface has a global coordinate chart $(U, \mathbf{x})$ with $U = \mathbb{R}^2$ and $\mathbf{x}(u,v) = (u, v, h(u,v))$.

(a) Show that this coordinate chart is regular for all $(u,v) \in U$. [5]

(b) State a formula for the Gaussian curvature in terms of E, F, G and L, M, N where

$$\begin{aligned} E &= \mathbf{x}_u \cdot \mathbf{x}_u \,, \quad F = \mathbf{x}_u \cdot \mathbf{x}_v \,, \quad G = \mathbf{x}_v \cdot \mathbf{x}_v \,, \\ L &= \mathbf{N} \cdot \mathbf{x}_{uu} \,, \quad M = \mathbf{N} \cdot \mathbf{x}_{uv} \,, \quad N = \mathbf{N} \cdot \mathbf{x}_{vv} \,, \end{aligned}$$

and $\mathbf{N}$ is the unit normal vector. Show that the Gaussian curvature associated with a chart of the form $\mathbf{x}(u,v) = (u, v, h(u,v))$ is

$$K = \frac{h_{uu}h_{vv} - h_{uv}^2}{(1 + h_u^2 + h_v^2)^2} \,.$$

[12]

(c) Find all points in (u,v) space where the surface is *parabolic*: i.e. the points where the Gaussian curvature vanishes. [8]

Question 3
Let $(U, \mathbf{x})$ be a regular coordinate chart for a surface $\mathcal{S} \subset \mathbb{R}^3$. Let

$$\boldsymbol{\gamma}(t) = \mathbf{x} \circ \mathbf{w}(t) := \mathbf{x}(w_1(t), w_2(t)) \,, \quad \alpha < t < \beta \,,$$

be a curve in $\mathcal{S}$ with $\mathbf{w}(t)$ a curve in U.

(a) What property does $\ddot{\boldsymbol{\gamma}}(t)$ have to satisfy for $\boldsymbol{\gamma}(t)$ to be a geodesic curve in $\mathcal{S}$? [4]

(b) Prove that geodesic curves have constant speed. [7]

(c) Define the *geodesic curvature* κ_g of a curve in $\mathcal{S}$. [5]

(d) Prove that a curve $\boldsymbol{\gamma}(t)$ in $\mathcal{S}$ is a geodesic curve if and only if $\boldsymbol{\gamma}(t)$ has constant speed and the geodesic curvature is zero. (Hint: express $\ddot{\boldsymbol{\gamma}}(t)$ in terms of a surface-fitted orthonormal frame.) [9]

Question 4

Let $f(x,y)$ be a smooth real-valued function satisfying $\|\nabla f\| \neq 0$, and suppose that the set $\{\,(x,y)\ :\ f(x,y)=0\,\}$ defines a curve in the $(x,y)-$plane.

(a) Write down the formula for the curvature of a plane parametrised curve $(x(t), y(t))$. [2]

(b) Assume that the curve satisfying $f(x(t), y(t)) = 0$ is parametrised by a solution of the differential equations

$$\dot{x} = -f_y \quad \text{and} \quad \dot{y} = f_x\,.$$

Prove that the curvature at any point on the curve can be expressed purely in terms of f and its derivatives by

$$\kappa = -\frac{1}{\|\nabla f\|^3}\det\begin{bmatrix} f_{xx} & f_{xy} & f_x \\ f_{yx} & f_{yy} & f_y \\ f_x & f_y & 0 \end{bmatrix}\,. \qquad (1)$$

where $\|\nabla f\| = \sqrt{f_x^2 + f_y^2}$. [9]

(c) Suppose the parametrisation is defined by the solution of the differential equation

$$\dot{x} = -h(t)\,f_y \quad \text{and} \quad \dot{y} = h(t)\,f_x\,,$$

where $h(t)$ is any nonzero function. Prove that the formula (1) for κ is independent of the choice of $h(t)$. [9]

(d) Compare the curve described in (b) with the curve in (c). What is the significance of the function $h(t)$ in part (c) ? [5]

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