# UNIVERSITY OF SURREY©

**B. Sc. Undergraduate Programmes in Mathematical Studies**

**Level HE3 Examination**

Module MS301 INTRODUCTION TO NONLINEAR PATTERNS

Time allowed – 2 hrs Spring Semester 2007

Attempt THREE questions
If a candidate attempts more than THREE questions only the best THREE questions will be taken into account.

**Question 1**

Consider the equations

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= -3x + \lambda x - 2x^2 - 2xy + 2\lambda x^2 - 6x^3, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -3x - y + \lambda x + x^2,\end{aligned}$$

which have a fixed point at $x = y = 0$.

(a) What is the value of $\lambda$ at which there is a bifurcation? [2]

(b) Show that to quadratic order the extended centre manifold that passes through the fixed point at the origin is given by

$$y = -3x + \lambda x + x^2.$$ [8]

(c) Find the equation for the evolution of $x$ on the extended centre manifold and hence deduce the type of bifurcation that occurs. [2]

(d) Find the fixed points of the equation in part c), and their stabilities and hence draw the bifurcation diagram using $\lambda$ as the horizontal coordinate. [9]

(e) If $x$ and $y$ are the concentrations of a blue and a colourless chemical respectively, dissolved in a glass of water, and $\lambda$ is the temperature of the water in degrees Celsius, what colour is the liquid in the glass at $1^o$C and at $4^o$C? What are the concentrations of the two chemicals at $4^o$C? [4]

**Question 2**

This question is about a steady bifurcation with $\mathcal{D}_2$ symmetry, the symmetry group of a rectangle.

(a) Write down the group elements in terms of two reflections $m_x$ and $m_y$. [2]

(b) Show that the group is Abelian (i.e. that every group element commutes with every other element) and hence that each element is in a conjugacy class on its own. [3]

(c) Show that the character table for $\mathcal{D}_2$ is

| | $C_1$ | $C_2$ | $C_3$ | $C_4$ |
|---|---|---|---|---|
| $R_1$ | 1 | 1 | 1 | 1 |
| $R_2$ | 1 | 1 | -1 | -1 |
| $R_3$ | 1 | -1 | 1 | -1 |
| $R_4$ | 1 | -1 | -1 | 1 |

where the $R_i$ are the four inequivalent irreps of $\mathcal{D}_2$ and the $C_i$ are its four conjugacy classes. Give your reasoning and state any theorems that you use. [6]

(d) Briefly explain how to use the equivariant branching lemma to deduce the symmetry of solution branches at the bifurcation. [2]

(e) Sketch each type of pattern guaranteed by the equivariant branching lemma, and state the isotropy subgroup of each one, giving your reasoning. [8]

(f) The form of the evolution equation for the eigenmode amplitude is the same in each case. Write it down, giving your reasoning, and deduce the type of bifurcation that gives rise to each pattern. [4]

**Question 3**

This question is about a steady bifurcation on a square lattice. A real scalar solution under the fundamental representation of the relevant symmetry group can be written to leading order in the form

$$u(x, y, t) = A(t)e^{ix} + B(t)e^{iy} + c.c., \quad A(t), B(t) \in \mathbb{C},$$

where $x$ and $y$ are Cartesian coordinates and $t$ is time.

(a) What is the symmetry group governing this bifurcation? [1]

(b) Write down the generators of the symmetry group, together with their actions on $\boldsymbol{x} = (x, y)$. [6]

(c) Write down an equation that defines the scalar action of the Euclidean group, E(2), on a function $v(\boldsymbol{x})$ and deduce the corresponding actions of the generators on $(A, B)$. [4]

(d) Use these to check that the following cubic order evolution equations for $A(t)$ and $B(t)$ are equivariant under the action of the symmetry group given in part a):

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}t} &= \mu A - |A|^2 A - a|B|^2 A, \\ \frac{\mathrm{d}B}{\mathrm{d}t} &= \mu B - |B|^2 B - a|A|^2 B,\end{aligned}$$

where $\mu$ is a real bifurcation parameter and $a$ is a real constant. [6]

(e) Assume now that the $x$-reflection symmetry is weakly broken. Explain why the amplitude equations can now be written

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}t} &= (\mu + \epsilon_1 + i\delta_1)A - (1 + \epsilon_2 + i\delta_2)|A|^2 A - (a + \epsilon_3 + i\delta_3)|B|^2 A, \\ \frac{\mathrm{d}B}{\mathrm{d}t} &= (\mu + \epsilon_4)B - (1 + \epsilon_5)|B|^2 B - (a + \epsilon_6)\beta|A|^2 B,\end{aligned}$$

where $\epsilon_j$, $\delta_j \in \mathbb{R}$, $|\epsilon_j| \ll 1$ and $|\delta_j| \ll 1$, $\forall j$. [4]

(f) Show that for $\mu + \epsilon_1 > 0$ there is a solution $A(t) = Re^{i\omega t}$, $B(t) = 0$, to the equations in part e), where

$$\begin{aligned}R^2 &= \frac{\mu + \epsilon_1}{1 + \epsilon_2}, \\ \omega &= \delta_1 - \delta_2 R^2,\end{aligned}$$

and describe what the corresponding pattern would look like. [4]

**Question 4**

This question is about a steady bifurcation on a hexagonal lattice.

(a) What is the symmetry group governing this bifurcation? [1]

(b) Write down the form of a solution on the lattice under the fundamental representation of the group. [2]

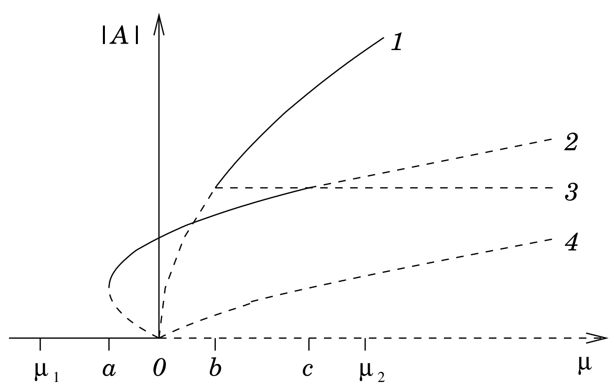
(c) Write down the isotropy subgroups that have one-dimensional fixed-point subspace under the fundamental representation, together with an orbit representative for each of them, and name the type of solution with that symmetry. [4]

(d) To leading order the amplitude equations for this bifurcation problem can be written

$$\begin{aligned}
\frac{\mathrm{d}A}{\mathrm{d}t} &= \mu A + \alpha \bar{B}\bar{C} - |A|^2 A - \beta(|B|^2 + |C|^2)A, \\
\frac{\mathrm{d}B}{\mathrm{d}t} &= \mu B + \alpha \bar{C}\bar{A} - |B|^2 B - \beta(|C|^2 + |A|^2)B, \\
\frac{\mathrm{d}C}{\mathrm{d}t} &= \mu C + \alpha \bar{A}\bar{B} - |C|^2 C - \beta(|A|^2 + |B|^2)C,
\end{aligned}$$

where $\mu$ is a real bifurcation parameter and $\alpha > 0$ and $\beta$ are real constants. Show that the solution $A = \sqrt{\mu}$, $B = C = 0$ for $\mu > 0$ is stable if $\beta > 1$ and $\sqrt{\mu} > -\alpha/(1-\beta)$. [10]

(e) The bifurcation diagram is given below, where solid lines represent stable solutions and dashed lines represent unstable solutions. What type of solutions are found on the branches labelled 1, 2, 3 and 4? Explain why branch 3 cannot be axial.




[3]

(f) You observe an experimental system where the solutions lie on a hexagonal lattice and correspond to the bifurcation diagram above. Initially the bifurcation parameter takes the value $\mu_1$. What would you expect to see as the bifurcation parameter is slowly increased to $\mu_2$ and then slowly decreased back to $\mu_1$? [5]

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