UNIVERSITY OF SURREY©

B. Sc. Undergraduate Programmes in Mathematical Studies

Level HE3 Examination

Module MS300 GALOIS THEORY

Time allowed – 2 hours

Spring Semester 2008

Answer any **three** of the five questions.

If you attempt more than three questions, only your BEST THREE answers will be taken into account.

Each question carries 30 marks.

Any results established in the course may be assumed and used without proof unless a proof is requested.

Question 1

Let $\alpha = 2^{1/3}$ and $\omega = \mathrm{e}^{2\pi\mathrm{i}/3}$.

(a) Give bases for $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$, for $\mathbb{Q}(\alpha, \omega)$ over $\mathbb{Q}(\alpha)$ and for $\mathbb{Q}(\alpha, \omega)$ over $\mathbb{Q}$. [6]

The polynomial $\mathrm{f} \in \mathbb{Q}[t]$ is defined by $\mathrm{f} = t^3 + 6t - 2$.

(b) Find the zeros of f in terms of α and ω. [8]

(c) Find $\Delta(\mathrm{f})$ and hence identify the Galois group $\Gamma_{\mathbb{Q}}(\mathrm{f})$. [2]

(d) Show that $\mathbb{Q}(\alpha, \omega)$ is the splitting field of f over $\mathbb{Q}$. [3]

(e) Define each element of $\Gamma_{\mathbb{Q}}(\mathrm{f})$ by its effect on α and on ω. [5]

(f) Sketch the lattice diagrams for this example, identifying each subgroup of $\Gamma_{\mathbb{Q}}(\mathrm{f})$ and subfield of $\mathbb{Q}(\alpha, \omega)$. [6]

Question 2

(a) If $\mathrm{f} = t^4 + ct^2 + dt + e \in \mathbb{Q}[t]$, it is known that

$$\mathrm{f} = \left(t^2 + kt + \frac{k^2 + c}{2} - \frac{d}{2k}\right)\left(t^2 - kt + \frac{k^2 + c}{2} + \frac{d}{2k}\right)$$

where $-k^2$ is a zero of ρ, the cubic resolvent of f.

(i) Letting α_1, α_2 be the zeros of the first factor and α_3, α_4 be the zeros of the second factor, show that if $u = (\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4)$ then $\rho(u) = 0$. [4]

You are given that the cubic resolvent of $t^4 + dt + e$ is $\rho = t^3 - 4et + d^2$.

(ii) If $\mathrm{f} = t^4 - 12t - 5$, show that -4 is a zero of the cubic resolvent of f. Use the above quadratic factors to find the zeros of f. [8]

(b) (i) Define the terms **content** of a polynomial and **primitive** polynomial in $\mathbb{Z}[t]$. [3]

(ii) Let f and g be primitive polynomials in $\mathbb{Z}[t]$. Prove that fg is also primitive. [6]

(c) Let K and L be fields with the property that every polynomial in $K[t]$ which has at least one zero in L has all its zeros in L.

(i) Give the name for a field extension $L : K$ with this property. [2]

(ii) Prove that L is a splitting field for some polynomial in $K[t]$. You may assume the Primitive Element Theorem. [5]

(iii) Name the extra properties that $L : K$ must have in order for it to be a **Galois extension**. [2]

Question 3

(a) Let f be a polynomial over a field K. State what is meant by saying that f is **irreducible** over K. [3]

(b) Give an example of a field extension $L : K$ and a polynomial $f \in K[t]$ such that f is irreducible over K but reducible over L. [3]

(c) State (do not prove) Eisenstein's criterion for the irreducibility of $f = \sum_{r=0}^{n} a_r t^r$ over $\mathbb{Z}$. [3]

(d) Let $f = 1 + t + t^2 + \cdots + t^{p-1}$ where p is prime. By considering $f(t+1)$, show that f is irreducible over $\mathbb{Q}$. You may assume that the binomial coefficient $\binom{p}{r}$ is divisible by p for $r = 1, \ldots, p-1$. [7]

(e) Deduce that $1 + t^p + t^{2p} + \cdots + t^{(p-1)p}$ is irreducible over $\mathbb{Q}$ for all primes p. [2]

(f) Show that $(1 - t^p)(1 + t^p + t^{2p} + \cdots + t^{(p-1)p}) = 1 - t^{p^2}$ and express $1 - t^{p^2}$ as a product of irreducible polynomials over $\mathbb{Q}$. [6]

(g) Hence show that when p is a prime greater than 2, a regular p^2-sided polygon cannot be constructed using ruler and compass only. [6]

Question 4

In this question f is the irreducible polynomial $t^4 - 4t^2 + 6$ in $\mathbb{Q}[t]$.

(a) By regarding f as a quadratic in t^2, show that $\alpha = \sqrt{2 + i\sqrt{2}}$ is one zero of f and find the other zeros of f. [5]

(b) If $\beta = \sqrt{2 - i\sqrt{2}}$, find $\alpha\beta$ in its simplest form. Deduce that $L = \mathbb{Q}(\alpha, \sqrt{6})$ is the splitting field of f over $\mathbb{Q}$. [5]

(c) By considering the minimal polynomials of α over $\mathbb{Q}$ and of $\sqrt{6}$ over $\mathbb{Q}(\alpha)$, find the degree of the extension $L : \mathbb{Q}$. [5]

(d) Let σ be the $\mathbb{Q}$-automorphism of L given by $\sigma(\alpha) = \beta, \sigma(\sqrt{6}) = -\sqrt{6}$. Show that $\sigma(\beta) = -\alpha$. Find the automorphisms σ^2, σ^3 and σ^4, defining each one by its effect on α and $\sqrt{6}$ [7]

(e) Deduce that the Galois group $\Gamma(L : \mathbb{Q})$ has a cyclic subgroup C of order 4. State an abstract group to which $\Gamma(L : \mathbb{Q})$ is isomorphic. [5]

(f) Let M be the subfield of L that is fixed by the elements of C. Identify a field extension whose Galois group is isomorphic to the quotient group $\dfrac{\Gamma(L : \mathbb{Q})}{C}$. [3]

Question 5

(a) (i) Let p be prime, $n \in \mathbb{N}$ and $q = p^n$. Let $\mathbb{F}_q$ be the finite field with q elements. Show that the map $\theta : \mathbb{F}_q \to \mathbb{F}_q$ defined by $\theta(x) = x^p$ is a field automorphism. [6]

(ii) Show that the polynomial $t^3 + t + 1$ is irreducible in $\mathbb{F}_2[t]$. [3]

Let α be a zero of $t^3 + t + 1$ in an extension field of $\mathbb{F}_2$.

(iii) State the value of q for which $\mathbb{F}_2(\alpha) \cong \mathbb{F}_q$. [2]

(iv) Identify the Galois group $\Gamma(\mathbb{F}_2(\alpha) : \mathbb{F}_2)$ and give an element of this group other than the identity. [3]

(b) What is meant by saying that a polynomial $\mathrm{f} \in \mathbb{Q}[t]$ is **solvable by radicals** over $\mathbb{Q}$? Give a condition on the Galois group $\Gamma_{\mathbb{Q}}(\mathrm{f})$ which is necessary and sufficient for f to be solvable by radicals over $\mathbb{Q}$. [4]

(c) Let $\mathrm{f} = t^5 - 80t + 20 \in \mathbb{Q}[t]$ and let $G = \Gamma_{\mathbb{Q}}(\mathrm{f})$.

Show that G contains an element of order 5 and a transposition. Stating any group-theoretic properties that you use, deduce that f is not solvable by radicals over $\mathbb{Q}$.

[12]

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