UNIVERSITY OF SURREY ©

B. Sc. Undergraduate Programmes in Mathematical Studies

Level HE2 Examination

Module MS238 STOCHASTIC PROCESSES

Time allowed – 2 hrs

Spring Semester 2008

Attempt THREE questions
If a candidate attempts more than THREE questions only the best THREE questions will be taken into account.

Question 1
Consider the standard gambler's ruin problem where player A competes against player B with stakes of £1 and each player has a finite amount of money. Let u_k denote the probability that A wins (i.e. B is ruined) starting with £k.

(a) Set up the usual recurrence relation for u_k and deduce that

$$u_k = \frac{1 - \left(\frac{q}{p}\right)^k}{1 - \left(\frac{q}{p}\right)^N}$$

when $p \neq \frac{1}{2}$. (Your answer should include an explanation of what N, p and q denote.) [8]

(b) Determine u_k in the remaining case $p = \frac{1}{2}$. [4]

(c) Compute the probability that B wins, and hence deduce that it is certain that the game will eventually terminate, for all values of p, k and N. [6]

(d) Now suppose that A starts with only £1. Player B believes that the chance of A winning is less than $\frac{1}{2}$ provided B starts with a sufficiently large amount of money. In fact this is not always correct. Take the appropriate limit to find the range of values of p for which B is correct. [7]

Question 2

(a) Consider a homogenous Markov chain with state space $\{1, 2, 3, 4\}$ and transition matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{7} & 0 & 0 & \frac{6}{7} \\ 0 & \frac{2}{7} & \frac{1}{7} & \frac{4}{7} \end{pmatrix}.$$

(i) Compute $P(X_3 = 1 | X_0 = 3, X_1 = 4, X_2 = 3)$. [2]

(ii) Compute $P(X_1 = 4, X_2 = 3, X_3 = 1 | X_0 = 3)$. [2]

(iii) Compute the probability of moving from state 4 to state 2 in two steps. [2]

(iv) Compute the expected time to absorption starting from state 3. [6]

(b) Let $\{X_n\}$ be a homogeneous Markov chain.

(i) When is a state j said to be *recurrent*? [2]

(ii) Write down a criterion for recurrence of state j in terms of the probabilities $p_{jj}^{(n)}$ of a return in n steps.

Write down a criterion for recurrence of state j in terms of the probabilities $f_{jj}^{(n)}$ of the first return in n steps.

Indicate which one of these two criteria is immediate from the definitions. [3]

(iii) Prove that $p_{jj}^{(n)} = \sum_{k=1}^{n} f_{jj}^{(k)} p_{jj}^{(n-k)}$ for $n \geq 1$. [4]

(iv) Set $f_{jj}^{(0)} = 0$ and let

$$F(t) = \sum_{n=0}^{\infty} f_{jj}^{(n)} t^n, \quad G(t) = \sum_{n=0}^{\infty} p_{jj}^{(n)} t^n.$$

Prove that

$$F(t)G(t) = G(t) - 1.$$

Hence prove that the two criteria in part (ii) are equivalent. [4]

Question 3

Let $\{X_n\}$ be a homogeneous Markov chain.

(a) Define the 1-*step transition probabilities* p_{ij} and the n-*step transition probabilities* $p_{ij}^{(n)}$, $n \geq 0$. [2]

(b) When are two states said to *communicate*? [2]

(c) Give the definition for a communicating class to be *closed*. [2]

(d) The transition matrix of a Markov chain with states $\{0, 1, 2, 3, 4, 5\}$ is given by

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 & \frac{3}{5} & \frac{2}{5} \end{pmatrix}$$

Determine the communicating classes and decide which classes are closed.

Determine (giving reasons) which states are recurrent, transient, periodic, aperiodic, and compute the period of any periodic states. [7]

(e) Consider the Markov chain with state space $\{1, 2, 3, 4\}$ and transition matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{5} & \frac{2}{5} & \frac{1}{5} & \frac{1}{5} \end{pmatrix}$$

(i) Explain why the Markov chain is irreducible and aperiodic. [2]

(ii) State a theorem from which it follows that there is a unique stationary distribution for this Markov chain. [2]

(iii) Compute the stationary distribution. [5]

(iv) Given the initial distribution $(0 \; \frac{1}{3} \; \frac{1}{3} \; \frac{1}{3})$ compute $\lim_{n\to\infty} P(X_n = 1)$, explaining your reasoning. [3]

Question 4
Let $\{X(t), t \geq 0\}$ be a stochastic process with state space $\{0, 1, 2, \ldots\}$ and $X(0) = 0$.

(a) State the mathematical properties that $X(t)$ must satisfy for the process to be a Poisson process with rate parameter $\lambda > 0$. [4]

(b) Let $X(t)$ be as in part (a).

(i) Derive the differential equations

$$\frac{dp_n(t)}{dt} = -\lambda p_n(t) + \lambda p_{n-1}(t)$$

for the probabilities $p_n(t) = P(X(t) = n)$, $n \geq 1$, and write down the differential equation for $p_0(t)$. [7]

(ii) Write down the appropriate initial conditions when $n = 0$. [2]

(iii) Solve for $p_0(t)$ and $p_1(t)$, and show that $p_2(t) = \frac{1}{2}\lambda^2 t^2 e^{-\lambda t}$. [7]

(c) Suppose that $\lambda = \frac{1}{3}$. Use the expressions for $p_1(t)$ and $p_2(t)$ to compute (i) $P(X(20) = 2)$ and (ii) $P(X(60) = 3 \,|\, X(20) = 1)$. [5]

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