UNIVERSITY OF SURREY©

B. Sc. Undergraduate Programmes in Mathematical Studies

Level HE2 Examination

Module MS238 STOCHASTIC PROCESSES

Time allowed – 2 hrs Autumn Semester 2006

Attempt THREE questions.
If a candidate attempts more than THREE questions only the best THREE questions will be taken into account.

Question 1

a) (Duration of gambler's game) Player A plays against player B. Player A starts with £k and player B with £$(N-k)$. In each play A wins £1 with probability $p>0$ and loses £1 with probability $q=1-p$. The game is over when either player has won all the money.

(i) Let $\Pr(d_k=n)$ be the probability that the game finishes after n plays. Justify the equation

$$\Pr(d_k=n)=p\Pr(d_{k+1}=n-1)+q\Pr(d_{k-1}=n-1).$$

[4]

(ii) Let $D_k=E(d_k)$ be the expected duration of the game when A starts with £k. Derive the equation

$$D_k=pD_{k+1}+qD_{k-1}+1$$

from part (i). [6]

(iii) Formulate boundary conditions D_0 and D_N and then solve the difference equation from (ii) for $q\neq p$. [9]

b) Consider the Markov chain on the state space $\{0,1,2,3\}$ whose transition probability matrix is given by

$$P=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.1 & 0.4 & 0.1 & 0.4 \\ 0.2 & 0.1 & 0.6 & 0.1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Starting in state 1, determine the probability that the Markov chain ends in state 0. [6]

Question 2

a) Define the mean recurrence time μ_i of a state, and explain what it means for a state to be transient, recurrent, positive recurrent and null recurrent. [4]

b) Consider a homogeneous Markov chain with state space $\mathbb{Z}$. Give a necessary and sufficient condition for state 0 to be recurrent in terms of the probabilities

$$\Pr(X_n = 0|X_0 = 0) = p_{00}^{(n)}.$$

[3]

c) Consider a random walk on the integers with transition probabilities

$$\begin{aligned} \Pr(X_{n+1} = k+1|X_n = k) &= p \\ \Pr(X_{n+1} = k-1|X_n = k) &= q \end{aligned}$$

where $p + q = 1$. Show that

$$p_{00}^{(2n)} = \frac{(2n)!}{n!n!} p^n q^n, \quad p_{00}^{(2n+1)} = 0.$$

[4]

d) Use Stirling's approximation

$$n! \sim n^n e^{-n} \sqrt{2\pi n}$$

to show

$$p_{00}^{(2n)} \sim \frac{(4pq)^n}{\sqrt{\pi n}}$$

and hence deduce that state 0 is recurrent if $p = q = \frac{1}{2}$. [4]

e) If $p \neq q$ show that $4pq < 1$ by discussing the function $f(x) = 4x(1-x)$ for $x \in [0,1]$. Hence show that state 0 is transient. [4]

f) For the Markov chain on the state space $\{1,2,3,4,5,6\}$ with the transition probability matrix

$$\begin{pmatrix} \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

determine which states are recurrent, null-recurrent, positive recurrent and which are transient. [6]

Question 3

a) Consider a Markov chain $\{X_n\}$ with state space $\{0, 1, \ldots, N\}$. Assume that the Markov chain starts in state i at time 0. Give an expression for the expected number of visits $M_j^{(n)}$ to state j up to time n, in terms of the m-step transition probabilities $p_{ij}^{(m)}$, $0 \leq m \leq n$. [4]

b) Show that an aperiodic, irreducible Markov chain on a finite state space $\{0, 1, \ldots, N\}$ on average spends a fraction of $1/\mu_j$ in state j where μ_j is the mean recurrence time of state j.
Hint: Compute the fraction of time spent in state j as $\lim_{n\to\infty} M_j^{(n)}/n$. Then use the Basic Limit Theorem and the fact that for a sequence $\{a_n\}_{n\in\mathbb{N}}$ with $a_n \to a$ we have $\frac{1}{n}\sum_{m=1}^{n} a_m \to a$. [8]

c) A Markov chain has transition probability matrix

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/3 & 1/3 & 1/3 \\ 1/2 & 0 & 1/2 \end{pmatrix}.$$

Compute the stationary distribution of this Markov chain. The costs incurred in spending one unit period are £4 in state 1, £1 in state 2 and £2 in state 3. What is the long run cost per unit period of this Markov chain? [7]

d) Determine the following limits in terms of the transition probabilities p_{ij} and the stationary distribution π of a finite state irreducible aperiodic recurrent Markov chain.

(i) $\lim_{n\to\infty} \Pr(X_{n+1} = j | X_0 = i)$.

(ii) $\lim_{n\to\infty} \Pr(X_n = k, X_{n+1} = j | X_0 = i)$.

(iii) $\lim_{n\to\infty} \Pr(X_n = k, X_{n-1} = j | X_0 = i)$.

Hint: Use the Basic Limit Theorem, the Markov property, and the fact that

$$\Pr(A \cap B | C) = \Pr(A | B \cap C) \Pr(B | C).$$

[6]

Question 4

Let $X(t)$ be the size of a population at time t and let $X(0) = N > 0$. We model $X(t)$ as a birth and death process. Let λ_k be the birth rate at population size k and let μ_k be the death rate at population size k.

a) State the postulates for a birth and death process. [3]

b) Let $P_n(t) = \Pr(X(t) = n)$. From the postulates for a birth and death process derive the differential equations

$$\begin{aligned} P_n'(t) &= -(\lambda_n + \mu_n)P_n(t) + \lambda_{n-1}P_{n-1}(t) + \mu_{n+1}P_{n+1}(t), \\ P_0'(t) &= -\lambda_0 P_0(t) + \mu_1 P_1(t). \end{aligned}$$

[8]

c) Assume that individuals in the population give birth at rate λ and die at rate μ. What are λ_n and μ_n under these assumptions? Explain your answer. [4]

d) Let $M(t) = E(X(t))$ be the expected population size at time t. Under the assumptions of part c) show that $M(t)$ satisfies the differential equation

$$M'(t) = (\lambda - \mu)M(t)$$

and solve it. State conditions on μ and λ under which the population is expected to become extinct. [5]

e) Let $X(0) = 2$ and let the process be a pure death process, i.e., $\lambda_k \equiv 0$. Let $\mu_1 = 1$, $\mu_2 = 2$. Compute the probability $P_0(t)$ that the population becomes extinct at time t. *Hint:* Argue first that $P_n(t) \equiv 0$ for $n > 2$. [5]

INTERNAL EXAMINER: C. WULFF
EXTERNAL EXAMINER: W. KRZANOWSKI