UNIVERSITY OF SURREY©

B. Sc. Undergraduate Programmes in Mathematical Studies
M. Math. Undergraduate Programmes in Mathematical Studies

Level HE2 Examination

Module MS237 MATHEMATICAL STATISTICS

Time allowed – 2 hours

Spring Semester 2008

Attempt THREE questions. If any candidate attempts more than THREE questions only the best THREE solutions will be taken into account.
Students may use approved calculators.
A formula sheet will be provided.
Cambridge Statistical Tables will be provided.

Question 1

Random variables X and Y have the joint probability density function

$$\begin{aligned} f_{X,Y}(x,y) &= k \qquad x^3 < y < x, \quad 0 < x < 1, \quad 0 < y < 1; \\ &= 0 \qquad \text{elsewhere.} \end{aligned}$$

(a) Sketch the region in which $f_{X,Y}(x,y)$ is non-zero and show that $k = 4$. [4]

(b) Show that the marginal density function of X is

$$f_X(x) = 4x(1-x^2) \qquad 0 < x < 1,$$

and find the marginal density function of Y. Hence, or otherwise, find non-zero constants c_1 and c_2 such that $c_1\,E[X] + c_2\,E[Y] = 0$. [8]

(c) State whether X and Y are independent, giving a reason for your answer. [2]

(d) Show that the conditional density function of Y given $X = x$ is

$$f(y|X=x) = \frac{1}{x(1-x^2)} \qquad x^3 < y < x, \quad 0 < x < 1, \quad 0 < y < 1.$$

Hence, obtain $E(Y|X)$. [4]

(e) Prove that $E[E(V|U)] = E[V]$ for any two random variables U , V and verify that this result holds for random variables X , Y. [7]

Question 2

Let X be a discrete random variable with probability mass function (p_n). The probability generating function of X is defined by

$$P(z) = E(z^X) = \sum_n p_n z^n.$$

(a) (i) Prove that $E(X) = P'(1)$.

(ii) Prove that $\text{Var}(X) = P''(1) + P'(1) - \{P'(1)\}^2$.

(iii) Let X and Y be independent random variables with probability generating functions $P_X(z)$ and $P_Y(z)$. Prove that the probability generating function of $X + Y$ is given by $P_{X+Y}(z) = P_X(z)P_Y(z)$. [9]

(b) (i) Show that the probability generating function of the Geometric (π) distribution is

$$P(z) = \frac{\pi z}{1 - (1 - \pi)z}.$$

(ii) Hence, or otherwise, prove that the Geometric (π) distribution has mean π^{-1} and variance $(1 - \pi)\pi^{-2}$. [7]

(c) Let U and V be independent Geometric (π) random variables and let $W = U + V$.

(i) Show that if u, n are positive integers such that $u \leq n$ then

$$P\{(U = u) \cap (W = n)\} = P(U = u)P(V = n - u).$$

(ii) Write down the probability generating function of W and find an expression for $P(W = n)$.

(iii) Hence or otherwise, show that the conditional probability mass function of U given W is

$$P\{(U = u) \mid (W = n)\} = \frac{1}{n - 1}, \quad \text{for } u = 1, \cdots, n - 1.$$

(iv) Consider two fair dice, one red and one blue. The blue die is thrown until a 1 is scored and then the red die is thrown until it scores a 2. Given that the total number of throws is 10, obtain the expectation of the number of throws with the red die. [9]

Question 3

Suppose that the random vector $\underline{X} = (X_1, \ldots, X_p)^T$ has mean vector $\underline{\mu} = (\mu_1, \ldots, \mu_p)^T$ and covariance matrix $\Sigma = (\sigma_{ij})$, and assume that $\underline{a} = (a_1, \ldots, a_p)^T$ and $\underline{b} = (b_1, \ldots, b_p)^T$ are p-vectors of real-valued non-random coefficients.

(a) Prove that

(i) $E(\underline{a}^T \underline{X}) = \underline{a}^T \underline{\mu}$; [3]

(ii) $\text{Var}(\underline{a}^T \underline{X}) = \underline{a}^T \Sigma\ \underline{a}$; [3]

(iii) $\text{Cov}(\underline{a}^T \underline{X}, \underline{b}^T \underline{X}) = \underline{a}^T \Sigma\ \underline{b}$. [3]

(b) Let X_1, X_2 and X_3 be independent random variables with zero means and variances σ_1^2, σ_2^2 and σ_3^2 respectively.

The random variables Y_1, Y_2 and Y_3 are defined by

$$\begin{aligned} Y_1 &= X_1 + X_2, \\ Y_2 &= X_1 - X_2, \\ Y_3 &= X_1 + 2X_2 + X_3. \end{aligned}$$

(i) Using the results in part (a), determine the covariance matrix of $\underline{Y} = (Y_1, Y_2, Y_3)^T$. Given that Y_1 and Y_2 are mutually independent, determine the ratio of the variances $\sigma_1^2 : \sigma_2^2$. Hence obtain the correlation between Y_1 and Y_3 and the correlation between Y_2 and Y_3. Comment on these values. [8]

(ii) Suppose further that X_1 and X_2 have the joint probability density function

$$\begin{aligned} f_{X_1,X_2}(x_1, x_2) &= e^{-(x_1+x_2)} \qquad 0 < x_1 < \infty, \quad 0 < x_2 < \infty; \\ &= 0 \qquad \text{elsewhere.} \end{aligned}$$

Obtain the probability density function of Y_1. [8]

Question 4

(a) The random variable X has the Beta(α, β) probability density function.

(i) Show that $\mathrm{E}[X] = \frac{\alpha}{\alpha+\beta}$ and $\mathrm{Var}[X] = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$. [6]

(ii) Show that the mode of the distribution of X is $\frac{\alpha-1}{\alpha+\beta-2}$, for $\alpha+\beta > 2$. [3]

(b) The random variable Y has probability density function

$$\begin{aligned} f_Y(y) &= \frac{1}{B(\gamma,\delta)} \frac{y^{\gamma-1}}{(1+y)^{\gamma+\delta}} \qquad 0 < y < \infty; \\ &= 0 \qquad \text{elsewhere.} \end{aligned}$$

Find and identify the distribution of $Z = \dfrac{1}{1+Y}$. [8]

(c) (i) Let X be a continuous random variable with distribution function $F_X(x)$. Write down the probability density function of the random variable $Y = F_X(X)$. [1]

(ii) Obtain the distribution function for the Beta$(\gamma, 1)$ distribution. [4]

(iii) The following random sample of size 5 is obtained from the $U(0,1)$ distribution:

$$0.904,\ 0.766,\ 0.384,\ 0.407,\ 0.739.$$

Using this sample and results from (c)(i) and (c)(ii), generate a random sample of size 5 from the Beta$(3,1)$ distribution. [3]

Question 5

(a) Prove the Cauchy-Schwartz inequality:

$$\{E(X_1X_2)\}^2 \leq E(X_1^2)E(X_2^2),$$

for random variables X_1 and X_2, with finite variances. [6]

(i) Suppose further that X_1 is a positive random variable, so $P(X_1 \leq 0) = 0$. Prove that

$$E\left(\frac{1}{X_1}\right) \geq \frac{1}{E(X_1)}.$$

[3]

(ii) Prove that $-1 \leq \text{Corr}(X_1, X_2) \leq 1$. [3]

(b) The joint probability mass function of the discrete random variables Y_1 and Y_2 is

$$\begin{aligned} p_{(Y_1,Y_2)}(y_1, y_2) &= \frac{y_1 y_2}{36} \quad \text{for} \quad y_1 = 1,2,3 \quad \text{and} \quad y_2 = 1,2,3; \\ &= 0 \qquad\quad \text{elsewhere.} \end{aligned}$$

(i) Find the joint probability mass function of $X_1 = Y_1Y_2$ and $X_2 = Y_2$. [6]

(ii) Find the marginal probability mass function of X_1 and compute $E(X_1)$. [4]

(iii) Using results from part (a), find a lower bound for $E\left(\frac{1}{2Y_1Y_2+3}\right)$. [3]

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