UNIVERSITY OF SURREY©

B. Sc. Undergraduate Programmes in Mathematical Studies

Level HE2 Examination

Module MS224 FUNCTIONS OF A COMPLEX VARIABLE

Time allowed – 2 hrs Spring Semester 2007

Attempt THREE questions.
If a candidate attempts more than THREE questions only the best THREE questions will be taken into account.

Question 1

a) What does it mean for a complex function $f(z)$ to be differentiable at $z_0 \in \mathbb{C}$? [3]

b) Suppose that

$$f(x+iy) = u(x,y) + iv(x,y)$$

is differentiable at $z_0 = x_0 + iy_0 \in \mathbb{C}$. Show that the partial derivatives

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$$

all exist at (x_0, y_0) and satisfy the Cauchy-Riemann equations.
Hint: Compute the difference quotient of

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

for Δz real and Δz purely imaginary and consider the limit $\Delta z \to 0$. [9]

c) Show that if f is complex differentiable in a domain $D \subseteq \mathbb{C}$ and either its real part $\text{Re}(f)$ or its imaginary part $\text{Im}(f)$ is a constant on D then f is constant on D. [6]

d) Let

$$f(z) = \begin{cases} (x^{4/3}y^{5/3} + ix^{5/3}y^{4/3})/(x^2+y^2) & \text{if} \quad z \neq 0 \\ 0 & \text{if} \quad z = 0 \end{cases}$$

Show that the Cauchy-Riemann equations hold at $z = 0$. Is $f(z)$ differentiable at $z = 0$?
Hint: Consider the difference quotient $f(\Delta z)/\Delta z$ for $\Delta z \to 0$ along the real axis and along the line $y = x$. [7]

Question 2

a) (i) Let two functions $P(x, y)$ and $Q(x, y)$ be continuous and possess continuous first partial derivatives in some simply-connected region D of the (x, y)-plane. Let γ be a closed curve in D, which encloses an area A. Under these assumptions state the two-dimensional form of Green's Theorem (the two-dimensional form of the Divergence Theorem). [4]

(ii) For an analytic function $f(z) = u(z) + iv(z)$ with continuous derivative f' prove Cauchy's Theorem

$$\int_\gamma f(z)\mathrm{d}z = 0$$

by splitting this integral into its real and imaginary part and applying Green's Theorem. [5]

b) (i) Show that

$$I := \int_0^{2\pi} \frac{d\theta}{4\cos\theta + 5} = -i \int_\gamma \frac{dz}{2z^2 + 5z + 2}$$

where γ is the path $z = e^{i\theta}$, $0 \leq \theta \leq 2\pi$. [3]

(ii) Decompose

$$f(z) = \frac{1}{2z^2 + 5z + 2}$$

into partial fractions. [5]

(iii) Using Cauchy's Theorem, the Contour Deformation Theorem and explicit calculation only, evaluate the integral of each partial fraction along the path γ defined in part (i). Hence show that

$$I = \frac{2\pi}{3}.$$

[8]

Question 3

a) Assume that $|z-z_0| < r < R$ and let ζ lie on a circle around z_0 with radius r. Compute the Taylor series of $g(z) = \frac{1}{z-\zeta}$ around $z = z_0$ and show that it is convergent. [4]

b) Suppose that $f(z)$ is differentiable in the disk $|z - z_0| < R$. Show that for all z with $|z - z_0| < R$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \text{ where } a_n = \frac{f^{(n)}(z_0)}{n!}.$$

Hint: Use the Cauchy Integral formula

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta$$

where γ is the circle $\gamma = \{\zeta \in \mathbb{C}, |\zeta - z_0| = r\}$ traversed in the positive sense and use part a). [5]

c) Let Γ be the circle $|z| = 2$ traversed in the positive sense. Use the Cauchy Integral formula and the Cauchy Integral formula for derivatives to compute

(i) $\int_\Gamma \frac{\sin 3z}{z-\frac{\pi}{2}} dz$ [1]

(ii) $\int_\Gamma \frac{ze^z}{2z-3} dz$ [2]

(iii) $\int_\Gamma \frac{e^{-z}}{(z+1)^2} dz$ [2]

(iv) $\int_\Gamma \frac{\sin z}{z^2(z-4)} dz$ [3]

d) (i) Given a Laurent expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{m=1}^{\infty} b_m (z - z_0)^{-m}$$

in the annulus $R_1 < |z - z_0| < R_2$, give an expression for the coefficients a_n and b_m which generalizes Cauchy's Integral formula.

[3]

(ii) Let $t \in \mathbb{R}$. Show that for $|z| > 0$ the following Laurent expansion holds true:

$$\exp\left\{\frac{t}{2}\left(z - \frac{1}{z}\right)\right\} = \sum_{n=-\infty}^{\infty} J_n(t) z^n,$$

where

$$J_n(t) = \frac{1}{2\pi i} \int_\gamma e^{\frac{t}{2}(z-\frac{1}{z})} z^{-n-1} dz = \frac{1}{\pi} \int_0^\pi \cos\left(t \sin\theta - n\theta\right) d\theta$$

and γ is the unit circle $|z| = 1$ traversed in the positive direction. [5]

Question 4

a) Let f be analytic in a domain D except at z_0, so that z_0 is an isolated singularity. What does it mean for z_0 to be a pole of order m, to be a removable singularity and to be an essential singularity? [6]

b) Define the residue of an isolated singularity. [3]

c) Let γ be the circle $|z| = 5$ traversed in the positive sense. Evaluate the integral

$$\int_\gamma \frac{\sin(z)}{z^2 - 4} dz$$

by means of the Cauchy Residue Theorem. [5]

d) Let f have an isolated singularity at z_0. Show that the residue of the derivative f' at z_0 is equal to zero. [5]

e) Given that

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

integrate e^{-z^2} around a rectangle with vertices at $z = 0, \rho, \rho + \lambda i$ and λi (with $\lambda > 0$). Take the real part of the integrals you obtain and let $\rho \to \infty$ to derive

$$\int_0^\infty e^{-x^2} \cos(2\lambda x) dx = \frac{\sqrt{\pi}}{2} e^{-\lambda^2}.$$

[6]

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