UNIVERSITY OF SURREY©

B. Sc. Undergraduate Programmes in Mathematical Studies
M. Math. Undergraduate Programmes in Mathematical Studies

Level HE2 Examination

Module MS221 OPERATIONS RESEARCH AND OPTIMIZATION

Time allowed – 2 hrs Spring Semester 2007

Answer any **three** of the four questions

If you attempt more than three questions, only your BEST THREE answers will be taken into account.

Each question carries 30 marks.

Question 1

A company makes three different lamps: types A, B and C.

Type A is made from 1 kg of metal and 3 kg of glass.
Type B is made from 1 kg of metal, 1 kg of glass and 2 kg of plastic.
Type C is made from 1 kg of metal and 1 kg of plastic.

The company makes a profit of £4, £5 and £3 respectively on each type A, type B and type C lamp that is sold.

For the next month's production there is at most 550 kg of metal, 300 kg of glass and 650 kg of plastic available. The company wants to maximize the profit it can get by making and selling x_1 type A, x_2 type B and x_3 type C lamps.

(a) Formulate the above information as a linear programming problem. [5]

(b) Use the simplex algorithm to find the optimal production plan. [10]

(c) Write down the matrices B and B^{-1} and the vector $\mathbf{b}$ such that the optimal numbers of lamps are given by the entries of $\mathrm{B}^{-1}\mathbf{b}$. [4]

(d) Write down a vector $\mathbf{c}$ such that the maximum profit is $\mathbf{c}^t\mathrm{B}^{-1}\mathbf{b}$. Verify that this expression is equal to the value that you found in part (b). [4]

(e) Suppose the profit on a type B lamp changes from £5 to £p.

Find the set of values of p for which the numbers of lamps found in part (b) are still optimal, and express the new maximum profit in terms of p. [7]

Question 2

(a) Consider the following primal-dual pair of linear programming problems:

$$\begin{array}{rl} \textbf{Primal:} & \text{Maximize } \mathbf{c}^t\mathbf{x} \text{ subject to } \mathrm{A}\mathbf{x} \le \mathbf{b} \text{ and } \mathbf{x} \ge \mathbf{0}, \\ \textbf{Dual:} & \text{Minimize } \mathbf{b}^t\mathbf{y} \text{ subject to } \mathrm{A}^t\mathbf{y} \ge \mathbf{c} \text{ and } \mathbf{y} \ge \mathbf{0}. \end{array}$$

Suppose $\mathbf{x}_0$ and $\mathbf{y}_0$ are feasible solutions of the primal and dual problems respectively. Prove that if ${\mathbf{x}_0}^t(\mathrm{A}^t\mathbf{y}_0 - \mathbf{c}) = {\mathbf{y}_0}^t(\mathbf{b} - \mathrm{A}\mathbf{x}_0) = 0$ then $\mathbf{x}_0$ and $\mathbf{y}_0$ are the optimal solutions of the primal and dual problems respectively. You may assume the weak duality theorem and its corollaries. [6]

(b) Consider the following linear programming problem:

$$\textbf{Minimize} \qquad w = 6y_1 + 8y_2 + y_3 + 2y_4$$

$$\textbf{subject to} \quad \left\{ \begin{array}{rcrcrcrcl} y_1 & + & 2y_2 & - & y_3 & & & \ge & 3 \\ 2y_1 & + & y_2 & + & y_3 & + & y_4 & \ge & 2 \end{array} \right.$$

$$\textbf{and} \qquad y_j \ge 0 \text{ for } j = 1, 2, 3, 4.$$

(i) Write down the dual of this problem. [4]

(ii) Solve the dual problem by a graphical method. [8]

(iii) Hence, *without* using the simplex algorithm, find the optimal solution of the given minimization problem. Give the minimum value of w and the corresponding values of y_1, y_2, y_3 and y_4. [7]

(iv) Suppose the second constraint is changed to $2y_1 + y_2 + y_3 + y_4 \ge k$, where $k < 2$. Find the smallest value of k for which $2y_1 + y_2 + y_3 + y_4 = k$ at the optimal solution. [5]

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Question 3

(a) Three factories F_1, F_2 and F_3 supply four supermarkets S_1, S_2, S_3 and S_4 with tea. F_1, F_2 and F_3 produce 8, 10, and 9 lorry-loads of tea per day respectively. S_1, S_2, S_3 and S_4 order 6, 5, 8, and 8 lorry-loads of tea per day respectively. The transportation costs (in pounds per lorry-load) are given in the following table, so for example it costs £7 to transport a lorry-load from F_2 to S_2.

	S_1	S_2	S_3	S_4
F_1	6	4	5	6
F_2	3	7	2	1
F_3	5	6	3	4

(i) Determine the minimum cost delivery plan, and find its cost. [10]

(ii) Because of an industrial dispute, F_3 will not supply S_1. Starting from your optimal solution to (i), find the new minimum cost. [6]

(b) A company's output, using non-negative quantities x, y and z of three inputs, is given by $P(x, y, z) = 25(xyz)^{1/5}$. This is to be maximized subject to the budget constraints $2x + y = 96$ and $y + 2z = 96$.

(i) Use the Lagrange Sufficiency Theorem to show that the maximum occurs when $x = y = z = 32$. You may assume that P is a concave function on $\mathbb{R}^3_+$. [10]

(ii) Use sensitivity analysis to estimate the new maximum output if the right-hand side of both constraints is increased from 96 to 100. [4]

Question 4

(a) Let f be a real-valued function defined on a convex open set $S \subset \mathbb{R}^n$.

(i) Define what it means for f to be a **convex function** on S. [3]

(ii) Prove that if f is convex on S and $\mathbf{x}^* \in S$ is a local minimizer of f, then $\mathbf{x}^*$ minimizes f globally over S. [9]

(iii) If f is convex on S, state a condition which must be satisfied by the Hessian matrix of f. [2]

(b) Let $f(x, y, z) = x^3 + 3y^2 + 2z^2 - 2xy - 2xz - 4y$.

(i) Show that f is a strictly convex function on the set

$$S = \left\{ (x, y, z) \in \mathbb{R}^3 \ : \ x > \frac{5}{18} \right\}.$$ [8]

(ii) Find, with justification, the minimum value of $f(x, y, z)$ over S. [8]

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