UNIVERSITY OF SURREY©

B. Sc. Undergraduate Programmes in Mathematical Studies
M. Math. Undergraduate Programmes in Mathematical Studies

Level HE2 Examination

Module MS221 OPERATIONS RESEARCH AND OPTIMIZATION

Time allowed – 2 hours

Autumn Semester 2007

Answer any **three** of the four questions

If you attempt more than three questions, only your
BEST THREE answers will be taken into account.

Each question carries 30 marks.

Question 1

A garden centre produces three types of fertiliser. There are 80 tons of nitrate and 90 tons of phosphate available.
To make 1000 bags of Regular Lawn fertiliser, 4 tons of nitrate and 2 tons of phosphate are needed.
To make 1000 bags of Super Lawn fertiliser, 4 tons of nitrate and 3 tons of phosphate are needed.
To make 1000 bags of Garden fertiliser, 2 tons of nitrate and 3 tons of phosphate are needed.

The profit, per 1000 bags of fertiliser, is £300 for Regular Lawn, £500 for Super Lawn and £400 for Garden. The total profit is to be maximized.

(a) Express this problem in linear programming form, stating what each of your variables represents. [4]

(b) Explain why it is not possible to find a basic feasible solution in which all three types of fertiliser are made. [3]

(c) Find the optimal production plan and state the maximum profit. [10]

The garden centre now decides to produce a fourth type of fertiliser, called Easy Grow. The available resources are unchanged.
To make 1000 bags of Easy Grow, 3 tons of nitrate and 2 tons of phosphate are needed. The profit per 1000 bags of Easy Grow is £100k.

(d) Find the largest value of k for which the solution in part (c) is still valid. [7]

(e) If $k = 4$, find the new optimal production plan and profit. [6]

Question 2

(a) Let $\mathbf{x}$ and $\mathbf{y}$ be two points in $\mathbb{R}^n$. State the name for an expression of the form $(1-r)\mathbf{x}+r\mathbf{y}$ where $0 \leq r \leq 1$. Give a geometrical interpretation of such an expression in $\mathbb{R}^2$. [4]

(b) Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be two feasible solutions of the linear programming problem

$$\text{Maximize } z = \mathbf{c}^t\mathbf{x} \text{ subject to } \mathrm{A}\mathbf{x} \leq \mathbf{b} \text{ and } \mathbf{x} \geq \mathbf{0}$$

where A is an $m \times n$ real matrix, $\mathbf{b} \in \mathbb{R}^m$ and $\mathbf{c}, \mathbf{x} \in \mathbb{R}^n$.

Let $L = \{(1-r)\mathbf{x}_1 + r\mathbf{x}_2 \; : \; 0 \leq r \leq 1\}$.

Show that all points in L are feasible for the problem. [8]

(c) **Either** by using the dual simplex algorithm **or** by solving the dual problem graphically, solve the linear programming problem

$$\textbf{Minimize} \qquad 2x_1 + 3x_2 + 8x_3$$

$$\textbf{subject to} \quad \begin{cases} -x_1 + 2x_2 + 6x_3 \geq -4 \\ \;\;\, x_1 + 3x_2 - 5x_3 \geq 9 \end{cases}$$

$$\textbf{and} \qquad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$$

Give the minimum value of z and the values of all main and surplus variables at the optimal point. [14]

(d) State the dual of the following linear programming problem:
Maximize $z = \mathbf{c}^t\mathbf{x}$ subject to $\mathrm{A}\mathbf{x} \leq \mathbf{b}$, where the components of $\mathbf{x}$ are unrestricted in sign. [4]

Question 3

(a) F_1, F_2 and F_3 are three factories, and $D_1, \ldots, D_5$ are five destinations. The supplies of a commodity available at F_1, F_2 and F_3 are 100, 160 and 140 units respectively. The quantities required at D_1, D_2, D_3, D_4, D_5 are 90, 60, 80, 100 and 70 units respectively.

The delivery costs per unit between the factories and destinations, in £, are as follows:

	D_1	D_2	D_3	D_4	D_5
F_1	9	3	6	7	4
F_2	7	5	2	10	6
F_3	5	4	9	8	10

The total delivery cost is to be minimized.

(i) Use the least-cost method to construct an initial basic feasible solution for this transportation problem. [3]

(ii) Starting from this initial solution, find the cheapest delivery plan. State the minimum cost. [11]

(iii) The cost of transporting one unit from F_2 to D_3 is increased by £a. Find the largest value of a for which the previous solution remains optimal. [4]

(b) (i) Define what is meant by a **concave function** from $\mathbb{R}^n$ to $\mathbb{R}$. [2]

(ii) Let $\mathrm{f}(x, y) = x^3 + 3xy^2$. Find the largest subset of $\mathbb{R}^2$ on which f is concave. [6]

(iii) Let $\mathrm{g}(\mathbf{x}) = \mathbf{b}^t\mathbf{x} + c$, where $\mathbf{b}$ is a constant vector in $\mathbb{R}^n$ and c is a real constant. Show that g is a concave function for all $\mathbf{x} \in \mathbb{R}^n$. [4]

Question 4

(a) State (but do not prove) the Lagrange Sufficiency Theorem for constrained minimization problems. [3]

(b) Use the method of Lagrange multipliers to solve the following problem:

Minimize $x_1 + x_2 + x_3$

subject to

$$\begin{cases} {x_1}^2 + x_2 = 3 \\ x_1 + 3x_2 + 2x_3 = 7. \end{cases}$$

Give a clear justification that the answer you find is a minimum. [12]

(c) Let $q(\mathbf{x}) = \mathbf{x}^t\mathrm{A}\mathbf{x}$, where A is an $n \times n$ real symmetric matrix and $\mathbf{x}$ is a variable vector in $\mathbb{R}^n$. State how the maximum value of $q(\mathbf{x})$ subject to the constraint $\mathbf{x}^t\mathbf{x} = 1$ can be found from the matrix A, and prove this result. [8]

(d) Use the result in (c) to determine the maximum value of $x^2 + 8xy + 7y^2$ subject to the constraint $x^2 + y^2 = 1$. Find values of x and y at which the minimum occurs. [7]

INTERNAL EXAMINER: Dr D.J. Fisher
EXTERNAL EXAMINER: Prof W. Krzanowski