UNIVERSITY OF SURREY©

B. Sc. Undergraduate Programmes in Mathematical Studies
M. Math. Undergraduate Programmes in Mathematical Studies

Level HE2 Examination

Module MS218 REAL ANALYSIS 2

Time allowed – 2 hrs Autumn Semester 2007

Attempt THREE questions
If a candidate attempts more than THREE questions only the best THREE questions will be taken into account.

Question 1

(a) Let $f : A \to \mathbb{R}$ and $x_0 \in A$. State and prove the property that relates $\lim_{x \to x_0} f(x)$ with $\lim_{x \to x_0^+} f(x)$ and $\lim_{x \to x_0^-} f(x)$. [5]

(b) Let f be a function defined on an open interval I, and $c \in I$. Prove that, if f is continuous at c and $f(c) > 0$, then there is an open interval $J \subset I$ such that $c \in J$ and $f(x) > 0$, for any $x \in J$. [6]

(c) Given that f and g are continuous functions from $\mathbb{R}$ to $\mathbb{R}$ and that $f(x) = g(x)$ for every rational number x, prove that $f = g$. [6]

(d) State Taylor's theorem. [4]

(e) Determine the second order Taylor polynomial of $\sqrt{x^2 + 1}$ about $x = -1$. [4]

Question 2

(a) State L'Hôpital's Rule. [3]

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Show that f is differentiable at $x = 0$ and find $f'(0)$. [5]

(c) Let f be as in part (b) and g be given by $g(x) = \sin x$. Show that

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = 0.$$

[4]

(d) State the Intermediate Value Theorem. [3]

(e) Suppose that $a_1 + a_2 + ... + a_n = 2$. Show that there is some $c \in (0, 1)$ such that $a_1 c + a_2 c^2 + ... + a_n c^n = 1$. [5]

(f) Suppose that f is continuous on the closed interval $[a, b]$, and differentiable on the open interval (a, b) and that $f'(x) = 0$ for all $x \in (a, b)$. Prove that f is constant on $[a, b]$. [5]

Question 3

(a) State the Mean Value Theorem for integrals. [3]

(b) Let $I = [a, b]$ and let f, g be continuous functions on I and such that

$$\int_a^b f = \int_a^b g.$$

Prove that there is a $c \in I$ such that $f(c) = g(c)$. [4]

(c) Prove that, if f is monotonically increasing on $[a, b]$ then f is Riemann integrable on $[a, b]$. [5]

(d) Let f be a twice differentiable function on an interval I. Show that if $f''(x) > 0$ for all $x \in I$, then f' is strictly increasing. [5]

(e) Suppose that f is defined on an open interval S and that f is twice differentiable with $f''(x) < 0$ for every $x \in S$. Suppose that $a, b \in S$ and that $a < b$. Let

$$g(x) = f(x) - f(a) - \left(\frac{f(b) - f(a)}{b - a}\right)(x - a)$$

for all $x \in [a, b]$.

(i) State Rolle's Theorem.

(ii) Prove that there exists a number $c \in (a, b)$ such that $g'(c) = 0$.

(iii) Prove that the function g' is strictly decreasing on the interval $[a, b]$.

(iv) Prove that $g(x) > 0$ for all $x \in (a, b)$. [8]

Question 4

(a) State the Fundamental Theorem of Calculus. [3]

(b) Let $I = [0, 1]$ and let $f : I \to \mathbb{R}$ be continuous. Suppose that

$$\int_0^x f = \int_x^1 f$$

for all $x \in I$. Prove that $f(x) = 0$ for all $x \in I$. [5]

(c) Let $I = [a, b]$ and let f, g, h be bounded functions on I to $\mathbb{R}$. Suppose that

$$f(x) \leq g(x) \leq h(x)$$

for all $x \in I$. Show that if f and h are integrable on I and if

$$\int_a^b f = \int_a^b h,$$

then g is also integrable on I and

$$\int_a^b f = \int_a^b g.$$

[6]

(d) Use the Mean Value Theorem to prove that $|\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$. [5]

(e) What does it mean to say that $\lim_{x\to\infty} f(x) = L$? [2]

(f) Show that, if $f : (a, \infty) \to \mathbb{R}$ is such that $\lim_{x\to\infty} xf(x) = L$ where $L \in \mathbb{R}$, then $\lim_{x\to\infty} f(x) = 0$. [4]

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