UNIVERSITY OF SURREY©

B. Sc. Undergraduate Programmes in Mathematical Studies

MMath Undergraduate Programmes in Mathematical Studies

Level HE2 Examination

Module MS218 REAL ANALYSIS 2

Time allowed - 2 hrs

Autumn Semester 2006

Attempt THREE questions
If a candidate attempts more than THREE questions only the best THREE questions will be taken into account.

Question 1

(a) Let $A \subset \mathbb{R}$. Define what it means to say that a function $f : A \to \mathbb{R}$ is uniform continuous in A. [3]

(b) Define $g : [0, \infty) \to \mathbb{R}$ to be $g(x) = \sqrt{x}$. Show that g is uniform continuous in $[0, \infty)$. You may use without proof that for a, $b \geq 0$, we have $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$, hence $\sqrt{a+b} - \sqrt{a} \leq \sqrt{b}$. [5]

(c) Define $h : [-1, 1] \to \mathbb{R}$ to be $h(x) = \sqrt{(\sin \pi x + x^3)^2}$. Which theorem will ensure that h has a minimum and a maximum on $[-1, 1]$? Verify all conditions of this theorem. Explain why the maximum of h will be in $(-1, 1)$, without differentiating h. [5]

(d) State the Mean Value Theorem. [3]

Let $A \subset \mathbb{R}$. A function $f : A \to \mathbb{R}$ is called *Lipschitz continuous* at $x_0 \in A$ if there is a $K > 0$ and some $\delta > 0$ such that $|f(x) - f(x_0)| \leq K\,|x - x_0|$ for all $x \in A$ with $0 < |x - x_0| < \delta$.

(e) Show that if f is differentiable for every $x_0 \in A$ with f' a continuous function, then f is Lipschitz continuous at every $x_0 \in A$. [5]

(f) Show that $g : [0, \infty) \to \mathbb{R}$ with $g(x) = \sqrt{x}$ is Lipschitz continuous at any $x_0 > 0$, but not Lipschitz continuous at $x_0 = 0$. [4]

Question 2

(a) Define $f : (0, \infty) \to \mathbb{R}$ to be $f(x) = \ln x$. Using the definition of a derivative, show that $f'(x) = \frac{1}{x}$ for $x > 0$. You may use without proof that $\lim\limits_{y \to 0} \frac{\ln(1+y)}{y} = 1$. [5]

(b) Let $g : \mathbb{R} \to \mathbb{R}$ be a four times differentiable function. What is the third order Taylor polynomial of g about $x = a$? [2]

(c) Define $g : (0, 1) \to \mathbb{R}$ as $g(x) = (2x^2 + 1)\ln(2x^2 - 1)$. Determine the second order Taylor polynomial about $x = 1$. [3]

(d) Show that every differentiable function is continous. [5]

Let $h : [-1, 1] \to \mathbb{R}$ be such that any derivative $h^{(m)}$, $m \in \mathbb{N}$, exists (i.e., h is infinitely differentiable) and there is some $C > 0$ such that

$$\left|h^{(m)}(x)\right| \leq Cm!\,|x|^m$$

for any $x \in [-1, 1]$ and $m \in \mathbb{N}$.

(e) Show that this implies that $h^{(m)}(0) = 0$ for any $m \in \mathbb{N}$ and determine the Taylor polynomial about $x = 0$ and the remainder of Taylor's Theorem (of order n, $n \in \mathbb{N}$) for this function. [5]

(f) Show that $h(x) = 0$, for any $x \in [-1, 1]$. [5]

Question 3

A function $f : \mathbb{R} \to \mathbb{R}$ is called *even* if $f(-x) = f(x)$ for all $x \in \mathbb{R}$ and is called *odd* if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

(a) Let $f : A \to \mathbb{R}$ be a function. What does it mean to say that the derivative $f'(x)$ exist for some $x \in A$ and what is $f'(x)$? [2]

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable odd function. Using the definition of a derivative, show that f' is an even function. [5]

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function for which f' is an even function. Does this imply that f is an odd function? Give a proof if this is the case or give a counter example if this is not the case. [3]

(d) Let $f : [a, b] \to \mathbb{R}$ be a bounded function and let D be a dissection of $[a, b]$. What is the upper sum $\mathcal{U}(f, D)$ and the lower sum $\mathcal{L}(f, D)$? [3]

Let $f : [a, b] \to \mathbb{R}$ be a bounded function, which is Riemann integrable. Define $g : [-b, -a] \to \mathbb{R}$ as $g(x) = f(-x)$ for $x \in [-b, -a]$.

(e) Let $D = \{x_0, \ldots, x_{n+1}\}$ be a dissection of $[a, b]$ and define the dissection $\widehat{D} = \{-x_{n+1}, \ldots, -x_0\}$ of $[-b, -a]$. Show that $\mathcal{U}(f, D) = \mathcal{U}(g, \widehat{D})$. [3]

(f) Using upper and lower sums or otherwise, show that $\displaystyle\int_a^b f = \int_{-b}^{-a} g$. [4]

(g) Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded function, which is Riemann integrable function on any interval $[a, b]$. Show that if f is odd, then the function $F(x) = \int_a^x f$ is even, for any $a \in \mathbb{R}$. [5]

Question 4

(a) Show that if $f : [a, b] \to \mathbb{R}$ is a bounded, Riemann integrable function, then $F : [a, b] \to \mathbb{R}$ with $F(x) = \displaystyle\int_a^x f$ is a continuous function. [4]

(b) Let $N \in \mathbb{N}$ and define $g : [-N, N] \to \mathbb{R}$ as $g(x) = 1$, if $x \neq 0$ and $g(0) = 0$. Using upper and lower sums or otherwise, show that g is Riemann integrable and find $\displaystyle\int_{-N}^{N} g$. [3]

(c) Define the function $f_1 : [0, 1] \to \mathbb{R}$ as

$$f_1(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [0, 1], \\ 0, & x \in \mathbb{R} \backslash \mathbb{Q} \cap [0, 1]. \end{cases}$$

(i) Show that $\mathcal{U}(f_1, D) = 1$ and $\mathcal{L}(f_1, D) = 0$ for any dissection D of $[0, 1]$.

(ii) Is f_1 Riemann integrable? If it is, find the integral $\displaystyle\int_0^1 f_1$ and otherwise, explain why it is isn't. [6]

(d) Define the function $f_2 : [0, 1] \to \mathbb{R}$ as

$$f_2(x) = \begin{cases} \frac{1}{q}, & x \in \mathbb{Q} \cap [0, 1], \text{ with } x = \frac{p}{q}, \text{ and } p,\ q \text{ no common divisors}, \\ 0, & x \in \mathbb{R} \backslash \mathbb{Q} \cap [0, 1]. \end{cases}$$

(i) Show that $\mathcal{L}(f_2, D) = 0$ for any dissection D of $[0, 1]$.

(ii) Let $D_4 = \{0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, 1\}$. Show that $\mathcal{U}(f_2, D_4) < \frac{1}{4}$.

(iii) For any $N \in \mathbb{N}$, find a dissection D_N of $[0, 1]$ such that $\mathcal{U}(f_2, D) < \frac{1}{N}$.

(iv) Show that f_2 Riemann integrable on any interval $[0, y]$ with $0 < y \leq 1$ and find $F_2(y) = \displaystyle\int_0^y f_2$.

(v) Show that $F_2(y)$ is differentiable and determine its derivative. How does this relate to the Fundamental Theorem of Calculus? [10]

(e) Is $g \circ f_2$ Riemann integrable? If it is, find the integral $\displaystyle\int_0^1 g \circ f_2$ and otherwise, explain why it is isn't. [2]

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