UNIVERSITY OF SURREY©

B. Sc. Undergraduate Programmes in Mathematical Studies
M. Math. Undergraduate Programmes in Mathematical Studies

Level HE2 Examination

Module MS217 Linear Partial Differential Equations

Time allowed – 2 hrs Spring Semester 2008

Attempt THREE questions
If a candidate attempts more than THREE questions only the best THREE questions will be taken into account.

Question 1

(a) Find the general solution of

$$\begin{aligned} u_t &= u_{xx}, \qquad 0 < x < \ell, \quad t > 0 \\ u(0,t) &= 0 = u_x(\ell,t), \qquad t > 0. \end{aligned}$$

Briefly describe its limit as $t \to \infty$. [15]

(b) (i) State the definition of pointwise and uniform convergence for a series of functions, $\sum_{n=1}^{\infty} f_n(x)$ converging to $f(x)$, on an interval $[0,\ell]$. [3]

(ii) Consider a function $f(x)$ that is 2ℓ periodic, where both $f(x)$ and $f'(x)$ are piecewise continuous and which has exactly one jump discontinuity on each interval of length 2ℓ. What is the limit of the Fourier series of f? Does it converge to its limit uniformly, pointwise, or neither? [3]

(c) Show that the equation

$$2u_{xx} - 7u_{xy} - 4u_{yy} = 0, \qquad (x,y) \in \mathbb{R}^2$$

is hyperbolic and find its general solution. [4]

Question 2

(a) Compute the Fourier cosine series of the function

$$f(x) = \begin{cases} 0 & \text{if } 0 < x < 2 \\ 2 & \text{if } 2 < x < 3 \end{cases}$$

on the interval $(0,3)$. [7]

(b) Suppose that both u_1 and u_2 are solutions to

$$\begin{aligned} \triangle u &= g \qquad x \in D \\ u &= h \qquad x \in \partial D \end{aligned}$$

where g and h are given continuous functions, and D is a smooth domain in $\mathbb{R}^2$. Prove that $u_1 = u_2$. [7]

(c) Consider the equation

$$\begin{aligned} u_t &= u_{xx} + au_x + bu, \qquad x \in \mathbb{R}, \quad t > 0 \\ u(x,0) &= u_0(x), \end{aligned}$$

where $u_0(x)$ is a given continuous and rapidly decaying function and a and b are fixed constants. Use the Fourier transform to find an expression for the solution of this equation in terms of the initial data $u_0(x)$. Note: you do not need to evaluate the inverse Fourier transform that appears in the solution you get. [7]

(d) Recall that the solution to the heat equation

$$\begin{aligned} u_t &= u_{xx}, \qquad x \in \mathbb{R}, \quad t > 0 \\ u(x,0) &= u_0(x) \end{aligned}$$

is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4t}} u_0(y) \mathrm{d}y.$$

Prove that, if $|u_0(x)| < M$ for all $x \in \mathbb{R}$ for some constant M, then there exists a constant K such that

$$|u(x,t)| \leq K$$

for all $x \in \mathbb{R}$ and $t > 0$. [4]

Question 3

(a) Construct the general *bounded* solution of $\triangle u = 0$ on the exterior disk D given by

$$D = \{(r,\theta) : 0 \le \theta < 2\pi, \quad r > a\}.$$

Note that the Laplacian in polar coordinates is given by

$$\triangle u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.$$

[15]

(b) Suppose that u is a solution of

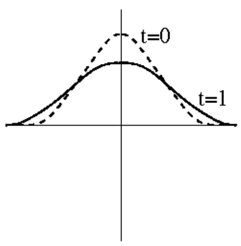
$$\begin{aligned} u_t &= u_{xx}, \qquad 0 < x < \ell, t > 0 \\ u(0,t) &= 0 = u(\ell,t), \quad t > 0 \\ u(x,0) &= g(x), \quad 0 < x < \ell, \end{aligned}$$

where $g(x) > 0$ for all $x \in [0,\ell]$. Prove that $u(x,t) \ge 0$ for all $t > 0$ and all $x \in (0,\ell)$. [6]

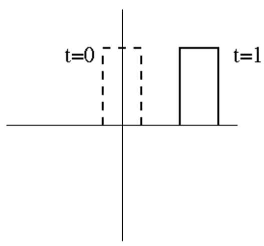
(c) Consider the following four figures, each containing a solution at time $t = 0$ (dotted curve) and $t = 1$ (solid curve).

(i) For each figure, state whether the solution could correspond to a solution of the heat equation. Justify your answer.

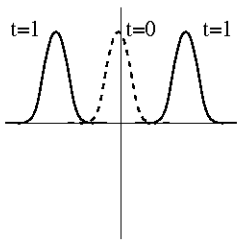
(ii) For each figure, state whether the solution could correspond to a solution of the wave equation. Justify your answer.



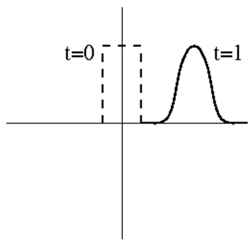
(i)



(ii)



(iii)



(iv)

[4]

Question 4

(a) (i) Find the general solution of

$$(x+1)u_x + 3u_y = 0 \tag{1}$$

and plot the characteristics. [7]

(ii) Find the solution of (1) that satisfies $u(x,0) = \sin(x+1)$. [2]

(iii) Can you prescribe values for $u(x,y)$ arbitrarily along the curve Γ, written below? Justify your answer.

$$\Gamma = \{(x,y) : x = y^2 - 1\}$$

[3]

(b) Consider the function $g(x) = x(1-x)$ on the interval $[0,1]$. Describe the convergence of its Fourier sine and cosine series on $(0,1)$. Which would have better convergence properties on $[0,1]$ and why? (Note: you do not need to compute the Fourier sine or cosine series for g.) [4]

(c) Suppose that $u(x,y)$ satisfies $\triangle u = 0$ inside the disk $x^2+y^2 < 1$ in $\mathbb{R}^2$, that u is continuous in $x^2+y^2 \leq 1$, and that on the boundary $x^2+y^2 = 1$

$$u(x,y) = 1 + 2y.$$

Calculate $u(0,0)$. [5]

(d) Prove that, if $u(x,t)$ is a solution to the heat equation with Dirichlet boundary conditions:

$$\begin{aligned} u_t &= u_{xx}, \qquad 0 < x < \ell, \quad t > 0 \\ u(0,t) &= 0 = u(\ell,t), \quad t > 0, \end{aligned}$$

then the energy

$$E(t) = \int_0^\ell u^2(x,t)\mathrm{d}x$$

is decreasing for all $t > 0$. [4]

INTERNAL EXAMINER: M. Beck
EXTERNAL EXAMINER: D. Chillingworth