UNIVERSITY OF SURREY©

B. Sc. Undergraduate Programmes in Mathematical Studies
M. Math. Undergraduate Programmes in Mathematical Studies

Level HE2 Examination

Module MS217 LINEAR PARTIAL DIFFERENTIAL EQUATIONS

Time allowed – 2 hrs Spring Semester 2007

Attempt THREE questions

If a candidate attempts more than THREE questions, only the best THREE questions will be taken into account.

Question 1

(a) State the mean-value property and the maximum principle for harmonic functions. [4]

(b) Suppose that u_1 and u_2 satisfy

$$\Delta u_j = 0 \qquad \text{for } x \in D$$

for $j = 1, 2$ with

$$u_1|_{\partial D} = g_1, \qquad u_2|_{\partial D} = g_2$$

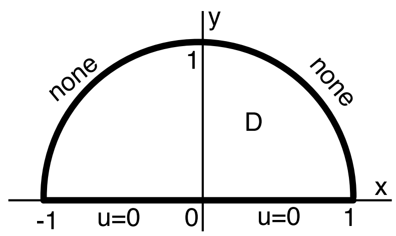
where g_1 and g_2 are given continuous functions with $g_1(x) \leq g_2(x)$ for all x, and D is a smooth domain in $\mathbb{R}^2$. Prove that

$$\max_{x \in D} |u_1(x) - u_2(x)| \leq \max_{x \in \partial D} |g_1(x) - g_2(x)|.$$ [6]

(c) Construct the general solution of $\Delta u = 0$ on the half disk D given by

$$D = \{(x, y) = (r \cos \varphi, r \sin \varphi);\ 0 < \varphi < \pi,\ 0 < r < 1\}$$

with boundary conditions as indicated here:



Note that the Laplace operator in polar coordinates is given by

$$\Delta u = u_{rr} + \frac{u_r}{r} + \frac{u_{\varphi\varphi}}{r^2}.$$ [15]

Question 2

(a) (i) Calculate the Fourier series of the function

$$f(x) = \begin{cases} 0 & 0 < x < 1 \\ 2 & 1 < x < 2 \end{cases}$$

extended with period 2 to the real line. [5]

(ii) What can you say about the convergence of the Fourier series, and its limit? [2]

(iii) Plot the function $f(x)$ together with its even and odd extensions on the interval $[-4, 4]$. [3]

(b) Prove that the sequence $f_n(x) := x^n$ converges uniformly to $f(x) = 0$ on the interval $[-\frac{1}{2}, \frac{1}{2}]$ as $n \to \infty$. [4]

(c) Use Fourier transform to find an expression for the solution of

$$\begin{aligned} u_t &= -u_{xxxx}, \qquad x \in \mathbb{R}, \quad t > 0 \\ u(x,0) &= u_0(x) \end{aligned}$$

where u_0 is a given continuous and rapidly decaying function. You do *not* need to explicitly compute the inverse Fourier transform of the solution you get. [7]

(d) The solution to the heat equation

$$\begin{aligned} u_t &= u_{xx}, \qquad x \in \mathbb{R}, \quad t > 0 \\ u(x,0) &= u_0(x) \end{aligned}$$

is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \exp\left(\frac{-(x-y)^2}{4t}\right) u_0(y)\,\mathrm{d}y.$$

For the initial condition

$$u_0(x) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

prove that

$$u(x,t) > 0 \qquad \forall x \in \mathbb{R}, \quad \forall t > 0.$$ [4]

Question 3

(a) (i) Find the general solution of

$$4u_x + 3yu_y = 0, \qquad (x, y) \in \mathbb{R}^2 \tag{1}$$

and plot the characteristics. [7]

(ii) Find the solution of (1) that satisfies $u(4/3, y) = y$. [2]

(iii) Can you prescribe values for $u(x, y)$ arbitrarily along the circle $\Gamma = \{(x, y);\ x^2 + y^2 = 1\}$? Justify your answer. [3]

(b) Show that the equation

$$2u_{xx} - 9u_{xt} - 5u_{tt} = 0, \qquad (x, t) \in \mathbb{R}^2$$

is hyperbolic and find its general solution. [7]

(c) Find the general solution of the equation [6]

$$u_x + u_y = u, \qquad (x, y) \in \mathbb{R}^2.$$

Question 4

(a) Find the general solution of

$$\begin{aligned} u_t &= u_{xx} - u, \qquad 0 < x < \ell, \quad t > 0 \\ u_x(0,t) &= 0 = u_x(\ell,t), \qquad t > 0. \end{aligned}$$

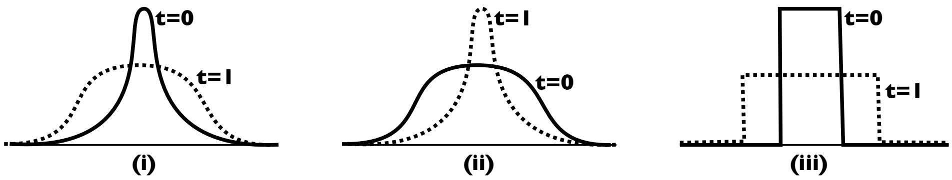
What is the limit of the general solution as $t \to \infty$? [15]

(b) Use the energy method to show that $u(x,t) = 0$ is the only solution to

$$\begin{aligned} u_t &= u_{xx} - u, \qquad 0 < x < 1, \quad t > 0 \\ u(0,t) &= 0 \\ u(1,t) &= 0 \\ u(x,0) &= 0. \end{aligned}$$

[6]

(c) Which of the graphs indicated below can correspond to a solution to the heat equation? (Multiple "yes" answers are possible; justify your answers): [4]



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