UNIVERSITY OF SURREY©

B. Sc. Undergraduate Programmes in Mathematical Studies
M. Math. Undergraduate Programmes in Mathematical Studies

Level HE2 Examination

Module MS216 FLUID MECHANICS

Time allowed – 2 hrs

Spring Semester 2006

Attempt THREE questions
If a candidate attempts more than THREE questions only the best THREE questions will be taken into account.

Question 1

(a) Give a brief explanation of the boundary conditions that apply where a fluid is in contact with a solid surface. You should deal with both viscous and inviscid fluids. [8]

(b) Use Cartesian coordinates to prove the following identities:

(i) $\nabla \times (\nabla f) = \mathbf{0}$;

(ii) $\nabla \cdot (\nabla \times \mathbf{F}) = 0$.

[5]

(c) Let V be a volume that is bounded by a solid simple closed surface ∂V. Use the Divergence Theorem,

$$\int_V \nabla \cdot \mathbf{F} \, \mathrm{d}\mathbf{x} = \int_{\partial V} \mathbf{n} \cdot \mathbf{F} \, \mathrm{d}S,$$

to show that

$$\int_V \nabla \times \mathbf{F} \, \mathrm{d}\mathbf{x} = \int_{\partial V} \mathbf{n} \times \mathbf{F} \, \mathrm{d}S.$$

[4]

(d) Show that if $\mathbf{u}$ is a potential flow of an inviscid fluid in V then

$$\int_{\partial V} \mathbf{n} \times \mathbf{u} \, \mathrm{d}S = \mathbf{0}.$$

Explain why this result does not contradict the idea that inviscid fluids can move along a solid boundary. [8]

Question 2

This question deals with steady two-dimensional unidirectional flow of an incompressible viscous fluid in a channel $\{(x, y, z) : y \in [-a, a], \ x, z \in \mathbb{R}\}$ with solid walls.

(a) State the Navier-Stokes equation for incompressible flow. Show that if the flow is $\mathbf{u} = u\mathbf{e_x}$ then u is a function of y only and the Navier-Stokes equation reduces to

$$-p_{,x} + \mu u_{,yy} = 0, \qquad p_{,y} = p_{,z} = 0.$$

[8]

(b) Show that if the channel walls are stationary and that the pressure gradient $G = -p_{,x}$ is a positive constant then

$$u(y) = \frac{G}{2\mu}\left(a^2 - y^2\right).$$

[5]

(c) Now suppose that the wall $y = a$ moves with velocity $U\mathbf{e_x}$, that the wall $y = -a$ moves with velocity $-U\mathbf{e_x}$, and that the pressure gradient is as in part (b). Calculate $u(y)$ and the volume flux (per unit width) in the x-direction, Q. Explain why Q is unaffected by the value of U. [7]

(d) Calculate the shear stress $\tau\mathbf{e_x}$ on the fluid at $y = a$ for the flow in part (c). Find the value of U for which τ is zero and sketch the velocity profile given that this value of U is chosen. [5]

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Question 3

(a) What condition must a flow $\mathbf{u}$ satisfy to be *irrotational*? State conditions under which an irrotational two-dimensional flow is a potential flow, i. e. $\mathbf{u} = \nabla\varphi$. [4]

(b) For a steady two-dimensional inviscid potential flow, the potential $\varphi(x, y)$ and the streamfunction $\psi(x, y)$ satisfy the Cauchy-Riemann equations:

$$\varphi_{,x} = \psi_{,y}\,, \qquad \varphi_{,y} = -\psi_{,x}\,.$$

Use this result to show that the complex potential $\Phi = \varphi + i\psi$ is a function of $\zeta = x + iy$ only. [4]

(c) Use cylindrical polar coordinates to calculate the inviscid flow $\mathbf{u} = \varphi_{,r}\,\mathbf{e_r} + \frac{1}{r}\,\varphi_{,\theta}\,\mathbf{e}_\theta$ that corresponds to the complex potential $\Phi = (a - ib)\ln\zeta$, where a and b are real constants. Sketch the streamlines, adding arrows to show the direction of flow, for

(i) $a = 0$ and $b > 0$;

(ii) $a < 0$ and $b = 0$;

(iii) $a < 0$ and $b > 0$. [12]

(d) Compare the flow in part (c)(iii) with a 'bathtub vortex,' which is formed when water drains from a bath and rotates around the plughole. What qualitative features do the two flows have in common? How do they differ? [5]

Question 4

(a) Explain the distinction between the partial time derivative $\frac{\partial}{\partial t}$ and the material derivative $\frac{D}{Dt}$. [4]

(b) Let $V(t)$ denote the bounded region of space that is occupied by a given blob of fluid at time t. Recall that if the position of a fluid particle in $V(t)$ at time t is $\mathbf{x}$, its position in $V(0)$ at time 0 is denoted by $\mathbf{X}$. Use the result that the Jacobian determinant

$$J(X_1, X_2, X_3, t) = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix}$$

satisfies $\frac{\partial J}{\partial t} = J\nabla \cdot \mathbf{u}$ to prove Reynolds' Transport Theorem for an arbitrary differentiable function $G(\mathbf{x}, t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} G(\mathbf{x}, t)\,\mathrm{d}\mathbf{x} = \int_{V(t)} \left\{ \frac{DG(\mathbf{x}, t)}{Dt} + G(\mathbf{x}, t)\nabla \cdot \mathbf{u} \right\} \mathrm{d}\mathbf{x}.$$

[6]

(c) Without assuming that the fluid is incompressible, use the fact that the mass of a given fluid blob is conserved to show that for an arbitrary differentiable function $F(\mathbf{x}, t)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho(\mathbf{x}, t)F(\mathbf{x}, t)\,\mathrm{d}\mathbf{x} = \int_{V(t)} \rho(\mathbf{x}, t)\frac{DF(\mathbf{x}, t)}{Dt}\,\mathrm{d}\mathbf{x}.$$

[7]

(d) The total *kinetic energy* of the fluid in the blob at time t is

$$E(t) = \int_{V(t)} \frac{\rho}{2}(\mathbf{u} \cdot \mathbf{u})\,\mathrm{d}\mathbf{x}.$$

Show that

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} = \int_{V(t)} \rho \left(\mathbf{u} \cdot \frac{D\mathbf{u}}{Dt} \right) \mathrm{d}\mathbf{x}.$$

(You may use Cartesian coordinates if you wish.) Assuming that the fluid in $V(t)$ is inviscid and incompressible, prove that $\frac{\mathrm{d}E(t)}{\mathrm{d}t}$ can be written as an integral over the boundary of $V(t)$. If the boundary is solid, how does the total energy change with time? [8]

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