UNIVERSITY OF SURREY©

B. Sc. Undergraduate Programmes in Mathematical Studies
M. Math. Undergraduate Programmes in Mathematical Studies

Level HE2 Examination

Module MS215 GROUPS AND SYMMETRY

Time allowed – 2 hrs

Spring Semester 2008

Attempt THREE questions
If a candidate attempts more than THREE questions only the best THREE questions will be taken into account.

Question 1

(a) State the conditions that must be satisfied by a binary operation $*$ on a set $\mathcal{S}$ for the pair $(\mathcal{S}, *)$ to form a group. [6]

(b) Prove that each element of a group has a *unique* inverse, indicating clearly which property of the binary operation you are using at each stage of the proof. [6]

(c) Which of the following sets are groups? For those that are groups show that all the conditions from part (a) are satisfied. For those that are not groups state which of the conditions are not satisfied. (You may use the facts that $(\mathbb{C}^*, \cdot)$ and $(\mathbb{R}, +)$ are groups)

(i) The set $\{1, -1, i, -i\}$, where $i = \sqrt{-1}$, with complex multiplication.

(ii) $(\mathbb{Z}_4^*, \cdot_4)$: the non-zero integers mod 4, with multiplication mod 4.

(iii) $(\mathbb{Q}^*, \div)$: the non-zero rational numbers, with division.

(iv) The set of all real $n \times n$ matrices, with matrix addition.

[13]

Question 2

In parts (b)-(d) of this question, $GL(n)$ is the group of real nonsingular $n \times n$ matrices with matrix multiplication and E is the $n \times n$ unit matrix.

(a) Suppose that H is a nonempty subset of a group G. Prove that H is a subgroup of G if and only if $xy^{-1} \in H,\ \forall x, y \in H$. [7]

(b) Let $H_n = \{A \in GL(n) : A^T A = E\}$. Prove that H_n is a subgroup of $GL(n)$. [6]

(c) Let $K_n = \{A \in GL(n) : AA^T = E\}$. Prove that $K_n = H_n$. [6]

(d) Let $P_n = \{A \in GL(n) : \det(\mathrm{A}) = \pm 1\}$. Prove that $H_n \subset P_n$. Show that $H_2 \neq P_2$ by finding an $A \in P_2 \setminus H_2$. [6]

Question 3

Let G and G' be groups, and suppose that $\phi : G \to G'$ is a homomorphism, so that $\phi(x)\phi(y) = \phi(xy),\ \forall x, y \in G$.

(a) Prove each of the following:

(i) if e, e' are the identity elements in $G,\ G'$ respectively then $e' = \phi(e)$;

(ii) if $x \in G$ then $(\phi(x))^{-1} = \phi(x^{-1})$;

(iii) if H is a subgroup of G and $H' = \{\phi(h) : h \in H\}$ then H' is a subgroup of G'. [10]

(b) Prove that if ϕ is an isomorphism, then ϕ^{-1} is a homomorphism. [6]

(c) The set $G' = \{1, 3, 7, 9\}$ forms a group under multiplication (modulo 10). Write down the group table for G'. Write down an isomophism between the cyclic group $\mathbb{Z}_4 =< r | \, r^4 = e >$ and G'. (You are not asked to prove that this is an isomorphism.) [9]

Question 4

In parts (b) and (c) of this question, G is an abelian group of order p^2, and p is a prime number.

(a) State (but do not prove) Lagrange's Theorem. [3]

(b) Show that the order of every element in $G \setminus \{e\}$ is either p or p^2. Explain why $G \cong \mathbb{Z}_{p^2}$ if and only if $\exists\, x \in G$ such that $| < x > | = p^2$. [7]

(c) Now suppose that G is not cyclic. Let $x \in G \setminus \{e\}$ and let $y \in G \setminus < x >$. Show that the elements $x^i y^j : i, j \in \{0, \cdots, p-1\}$ are distinct, and hence conclude that $G \cong \mathbb{Z}_p \times \mathbb{Z}_p$. (You may use the fact that $y^m \notin < x >$ for each $m \in \{1, \cdots, p-1\}$, without proving it.) [7]

(d) Determine all possible abelian groups of order 16, explaining why these are the only such subgroups. [8]

Question 5

In parts (c)-(e) of this question, suppose that H and J are normal subgroups of a group G, and that $H \subset J$.

(a) State (but do not prove) the First Isomorphism Theorem. [3]

(b) Using the additive group $(\mathbb{R}, +)$ and the multiplicative group $(\mathbb{C}^*, \cdot)$, show that $\mathbb{R}/\mathbb{Z} \cong S^1$, where S^1 is the unit circle. [4]

(c) Prove that $H \lhd J$. [3]

(d) Let $\phi : G/H \to G/J$ be defined by $\phi(xH) = xJ$ for all $x \in G$. Prove that ϕ is a surjective homomorphism, and calculate $\mathrm{Ker}(\phi)$. Hence, show that $J/H \lhd G/H$ and that $(G/H)/(J/H) \cong G/J$. [8]

(e) Now suppose that $G = \mathbb{D}_{12} =< r, s : r^{12} = s^2 = e, sr = r^{11}s >$. By choosing suitable subgroups H and J, use the results of part (c) to prove that $\mathbb{Z}_3 \lhd \mathbb{D}_6$ and $\mathbb{D}_6/\mathbb{Z}_3 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. State whether or not you could have chosen any other subgroups to prove this result. [7]

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