UNIVERSITY OF SURREY©

B. Sc. Undergraduate Programmes in Mathematical Studies

Level 2 Examination

Module MS215 GROUPS AND SYMMETRY

Time allowed - 2 hrs Spring Semester 2006

Attempt THREE questions. If any candidate attempts more than THREE questions only the best THREE solutions will be taken into account. The maximum attainable marks for each part of each question are indicated in square brackets [ ] .

Question 1

(a) State the conditions that must be satisfied by a binary operation $*$ on a set S for the pair $(S, *)$ to form a group. [6]

(b) Which of the following sets with binary operations are groups? For those that are groups show that all the conditions you listed in answer to (a) are satisfied. For those that are not groups state which of the conditions are not satisfied.

(i) $(\mathbb{Z}_6, +_6)$: the integers mod 6 with addition mod 6.

(ii) $(\mathbb{Z}_6^*, \cdot_6)$: the non-zero integers mod 6 with multiplication mod 6.

(iii) $(\mathbb{Z}, -)$: the integers with subtraction.

(iv) $(\mathbb{Q}^*, \cdot)$: the non-zero rational numbers, with multiplication. [15]

(c) Show that $(\mathbb{Z}_7^*, \cdot_7)$the non-zero integers mod 7 with multiplication mod 7, is a *cyclic* group. What is it generated by? [4]

Question 2

(a) Let G and G' be groups. What conditions must be satisfied by a map $\phi : G \to G'$ for it to be:

(i) a homomorphism;

(ii) an isomorphism. [5]

(b) The set of all integers that are divisible by the strictly positive integer n is written as $n\mathbb{Z}$ where $\mathbb{Z}$ is the set of integers.

(i) Prove that $n\mathbb{Z}$ is a subgroup of $\mathbb{Z}$ under addition. [4]

(ii) Define a map $\phi : n\mathbb{Z} \to \mathbb{Z}$ that is an injective homomorphism and prove that it has these properties. [5]

(iii) Define a map $\phi : n\mathbb{Z} \to \mathbb{Z}$ that is a homomorphism that is *not* injective. [4]

(iv) Define a map $\phi : n\mathbb{Z} \to \mathbb{Z}$ that is an isomorphism, and prove that it *is* an isomorphism. [7]

Question 3

(a) State the three defining properties of an equivalence relation on a set X. [6]

(b) Let H be a subgroup of the finite group G.

(i) Prove that the following relation $\sim$ on G is an equivalence relation:

$$x_1 \sim x_2 \text{ if and only if there exist } h_1 \in H,\ h_2 \in H \text{ and } g \in G$$
$$\text{such that } x_1 = gh_1 \text{ and } x_2 = gh_2.$$

[6]

(ii) Prove that all the equivalence classes of the relation defined in (a) have the same number of elements. [5]

(iii) Deduce that the number of elements in H divides the number in G, stating clearly any results that you use about equivalence relations. [4]

(iv) If the relation $x_1 \sim x_2$ defined in (i) holds, is there a *unique* $g \in G$ such that $x_1 = gh_1$ and $x_2 = gh_2$ for some $h_1 \in H$ and $h_2 \in H$? Explain your answer. [4]

Question 4

(a) Let $\phi : G \to G'$ be a homomorphism between groups.

(i) Show that the kernel K of ϕ is a normal subgroup of G. [4]

(ii) Show that the image I of ϕ is a subgroup of G'. [2]

(iii) What is the relationship between G, K and I? [2]

(b) Let $GL(n)$ denote the group of real nonsingular $n \times n$ matrices and

$$\begin{aligned} O(n) &= \left\{A \in GL(n) \ : \ A^T A = E\right\} \\ SO(n) &= \{A \in O(n) \ : \ \det(A) = 1\} \end{aligned}$$

where $\det(A)$ is the determinant of A, A^T is the transpose of A and E is the $n \times n$ identity matrix. In your answers to the following questions you may use properties of the determinant without proof provided you state them clearly.

(i) Show that $SO(n)$ is a normal subgroup of $O(n)$. [3]

(ii) What group is the quotient group $O(n)/SO(n)$ isomorphic to? Justify your answer. [5]

(c) Let M denote the set of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix}$$

where a is a non-zero real number, $a \in \mathbb{R}^*$, and b is any real number, $b \in \mathbb{R}$.

(i) Show that M is a group under matrix multiplication. [3]

(ii) Show that M has a normal subgroup N isomorphic to $(\mathbb{R}, +)$ such that the quotient group M/N is isomorphic to $(\mathbb{R}^*, \cdot)$ [6]

Question 5

(a) State (but do not prove) the Second Isomorphism Theorem. [4]

(b) The dihedral group

$$D_6 =< r, s \,|\, r^6 = s^2 = e,\ sr = r^5 s >$$

has a subgroup of order four, namely $J =< r^2 >$. Prove that J is a normal subgroup of D_6. [6]

(c) The dihedral group D_6 has a subgroup $H = \{e, r^3, s, r^3 s\}$. Show that H is not a normal subgroup of D_6. [3]

(d) Use the above results to show that $D_6/\mathbb{Z}_3 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. [Hint: note that $J \cong \mathbb{Z}_3$.] [4]

(e) Explain (giving reasons) which of the following are true:

(i) $D_6/(\mathbb{Z}_3 \times \mathbb{Z}_2) \cong \mathbb{Z}_2$;

(ii) $D_6/(\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \cong \{e\}$. [8]

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