UNIVERSITY OF SURREY©

B. Sc. Undergraduate Programmes in Mathematical Studies

Level HE2 Examination

Module MS213 ORDINARY DIFFERENTIAL EQUATIONS

Time allowed – 2 hrs Spring Semester 2007

Attempt THREE questions
If a candidate attempts more than THREE questions only the best THREE questions will be taken into account.

Question 1

(a) Define what it means for a function $x(t)$ to be a *solution* to the differential equation

$$\frac{dx}{dt} = f(x,t).$$

[3]

(b) Determine the values of the real constants a, b and c for which $x(t) = at^2 + bt + c$ is a solution to the differential equation

$$\left(\frac{dx}{dt}\right)^2 = 4x + 24t^2$$

[5]

(c) State an existence and uniqueness theorem for the initial value problem

$$\frac{dx}{dt} = f(x,t), \quad x(t_0) = x_0.$$

[4]

(d) Consider the ODE

$$\frac{dx}{dt} = x^3 - 25x$$

(i) Sketch the phase portrait. [3]

(ii) State which equilibria are attractors or repellors (or neither). [3]

(iii) Determine which equilibria are sinks or sources (or neither). [3]

(iv) Describe the asymptotic behaviour (as $t \to \pm\infty$ and/or $x(t) \to \pm\infty$) of the solution $x(t)$ satisfying $x(1) = 6$. [4]

Question 2

(a) Use the variation of parameters method to find the general solution to the ordinary differential equation

$$\frac{d^2x}{dt^2} - 7\frac{dx}{dt} = 8e^{7t}$$

[7]

(b) Consider the linear system of first order differential equations $\mathbf{x}' = A(t)\mathbf{x}$ where $A(t)$ is a 3×3 matrix.

(i) State conditions on the entries of $A(t)$ that guarantee that the initial value problem

$$\mathbf{x}' = A(t)\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0,$$

has a unique solution on the interval $J = (-1, 1)$ for all vectors $\mathbf{x}_0 \in \mathbb{R}^3$. [2]

(ii) Assume that your conditions in (i) are satisfied. By making suitable choices of $\mathbf{x}_0 \in \mathbb{R}^3$, construct three linearly independent solutions $\mathbf{z}_1(t)$, $\mathbf{z}_2(t)$, $\mathbf{z}_3(t)$ to $\mathbf{x}' = A(t)\mathbf{x}$.

Prove that $\mathbf{x}(t) = c_1\mathbf{z}_1(t) + c_2\mathbf{z}_2(t) + c_3\mathbf{z}_3(t)$ is the general solution. [7]

(iii) Find the general solution to the linear system

$$\mathbf{x}' = A(t)\mathbf{x}, \quad A(t) = \begin{pmatrix} -4 & 2 & 0 \\ 14 & -1 & 0 \\ 8 & 3 & 3 \end{pmatrix}$$

[9]

Question 3

Consider the nonlinear system $\mathbf{x}' = f(\mathbf{x})$ where $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^1.

(a) Let $\mathbf{x}_0 \in \mathbb{R}^n$. Prove that $\mathbf{x}(t) \equiv \mathbf{x}_0$ is a solution if and only if $f(\mathbf{x}_0) = \mathbf{0}$. [2]

(b) Define what it means for an equilibrium to be *asymptotically stable*. [4]

(c) Define *Lyapunov function* and *strict Lyapunov function*. Describe the consequence of the existence of such functions. [6]

(d) Compute a Lyapunov function for the equilibrium $(-1, 0, 1)$ for the system

$$\begin{aligned}\dot{x} &= -x^3 - 3x^2 - 3x + 5z^2 - 10z + 4\\ \dot{y} &= -13z + 7z^2 + 6\\ \dot{z} &= -4y - 2z + 2\end{aligned}$$

[7]

(e) Consider the system of equations

$$\begin{aligned}\dot{x} &= kx + 2y - (k+3)x^3\\ \dot{y} &= -4x - y^3\end{aligned}$$

where $k \in \mathbb{R}$ is a constant.

Determine the stability properties of the equilibrium $(x, y) = (0, 0)$ for each value of k. [6]

Question 4

(a) Consider the equilibrium $\mathbf{x} = \mathbf{0}$ for the linear system $\mathbf{x}' = A\mathbf{x}$ where A is a constant $n \times n$ matrix.

Define *hyperbolic*, *sink*, *source*, *saddle*. [4]

(b) Give an example that shows that an equilibrium for a linear system as in part (a) can be stable even if it is not a sink. [2]

(c) Sketch the phase portrait for the linear system

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{pmatrix} -5 & 2 \\ -1 & -2 \end{pmatrix}$$

[5]

(d) Reduce the second order nonlinear equation

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} + x^2 - \left(\frac{dx}{dt}\right)^2 = 1$$

to a system of first order equations, and compute the equilibria and their stability. [6]

(e) Sketch the local phase portraits near each equilibrium in part (d). [8]

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