UNIVERSITY OF SURREY©

B. Sc. Undergraduate Programmes in Mathematical Studies
M. Math. Undergraduate Programmes in Mathematical Studies

Level HE2 Examination

Module MS213 ORDINARY DIFFERENTIAL EQUATIONS

Time allowed – 2 hrs | Autumn Semester 2007

Attempt THREE questions

If a candidate attempts more than THREE questions only the best THREE questions will be taken into account.

Question 1

(a) Let x_0, $t_0 \in \mathbb{R}$ and let $f : \mathbb{R}^2 \to \mathbb{R}$. Consider the IVP

$$\frac{dx}{dt} = f(x,t), \quad x(t_0) = x_0.$$

(i) State an existence and uniqueness theorem for this IVP. [3]

(ii) Give an example of a function f that does not satisfy the conditions of the existence and uniqueness theorem and has non-unique solutions. Give two solutions of the initial value problem to illustrate the non-uniqueness. [5]

(b) Consider the ODE

$$\frac{dx}{dt} = (x-1)^2(x+2)(x-k),$$

where $k \geq 0$ is a non-negative real constant. The phase portrait will change if k varies.

(i) Find all possible types of phase portraits by varying $k \geq 0$ and give the range of k-values for which each type of phase portrait occurs. (Partial marks can be obtained for deriving the phase portrait for $k = 3$.) [6]

(ii) Take $k = 3$. Which solutions have the property that $x(t) \to 1$ for $t \to \infty$? [2]

(iii) Take $k = 3$. Which solutions are bounded for all time? [2]

(iv) Take $k = 3$. Describe the asymptotic behaviour ($|t| \to \infty$ and/or $|x| \to \infty$) of the solution $x(t)$ which satisfies $x(5) = -3$. [2]

(c) Consider the ODE

$$\frac{dx}{dt} = f(x)\,,$$

with $f : \mathbb{R} \to \mathbb{R}$ a function which satisfies the conditions of the Existence and Uniqueness Theorem. Let x_0 be such that $f(x_0) = 0$. Let $x(t)$ be a solution for $t \in I$ with I an open interval in $\mathbb{R}$. Show that if for some $t_0 \in I$, it holds that $x(t_0) > x_0$, then $x(t) > x_0$ for all $t \in I$. [5]

Question 2

(a) Let $x_1(t), \dots, x_k(t)$ be real functions (i.e. $x_i : \mathbb{R} \to \mathbb{R}$). Define what it means to say that the functions $x_1(t), \dots, x_k(t)$ are linearly independent. [3]

(b) Let $x_1,\, x_2 : \mathbb{R} \to \mathbb{R}$ be differentiable functions. Give the definition of the Wronskian of x_1 and x_2. [3]

(c) Consider the ODE

$$a_2(t)\,\frac{d^2x}{dt^2}(t) + a_1(t)\,\frac{dx}{dt}(t) + a_0(t)\,x(t) = 0,$$

with a_2, a_1, a_0 continuous functions on $\mathbb{R}$, $a_2(t) \neq 0$. Let $x_1(t)$ and $x_2(t)$ be solutions of the ODE and let $t_0 \in \mathbb{R}$. Show that x_1 and x_2 are linearly independent if and only if the Wronskian $W(t_0) \neq 0$.
You may use the Existence and Uniqueness Theorem without giving a proof. [7]

(d) Consider the ODE

$$t^2\,\frac{d^2x}{dt^2} + 4t(t-1)\,\frac{dx}{dt} + (6 - 8t + 3t^2)\,x = 0.$$

(i) Show that $x_1(t) = t^2\,e^{-t}$ is a solution of this ODE for $t > 0$. [3]

(ii) Using the method of variation of parameters, find a solution of this ODE that is linearly independent of $x_1(t)$. [6]

(iii) Give the general solution of this ODE. [3]

Question 3

(a) Consider the nonlinear system $\dot{\mathbf{x}} = f(\mathbf{x})$ where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable function. If $\mathbf{x}_0$ is an equilibrium of the nonlinear system, what does it mean to say that

(i) $\mathbf{x}_0$ is linearly stable;

(ii) $\mathbf{x}_0$ is nonlinearly stable? [4]

(b) Consider the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x - z^3 \\ \dot{z} &= -x^2 \sin z + y \end{aligned}$$

(i) Show that this system has an equilibrium at the origin. [2]

(ii) Find values a, b, such that $V(x, y, z) = a\,x^2 + b\,y^2 + z^4$ is a Lyapunov function for the equilibrium at the origin. [6]

(iii) From this Lyapunov function, can you conclude that the origin is stable or asymptotically stable? Explain your answer. [3]

(c) Let $A \in \mathbb{R}^{2\times 2}$ be a real matrix with eigenvalues λ_1 and λ_2. The eigenvector for the eigenvalue λ_1 is $\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and the eigenvector for λ_2 is $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

(i) If $\lambda_1 = -1$ and $\lambda_2 = 2$, sketch the phase portrait of the ODE $\dot{\mathbf{x}} = A\,\mathbf{x}$. [3]

(ii) Let $B \in \mathbb{R}^{2\times 2}$ be a constant coefficient real matrix with same eigenvectors as A. The eigenvalues of B are denoted by σ_1 and σ_2. Define the matrix C to be $C = AB$, hence C has eigenvalues $\lambda_1\sigma_1$ and $\lambda_2\sigma_2$. Suppose that both the ODEs $\dot{\mathbf{x}} = A\,\mathbf{x}$ and $\dot{\mathbf{y}} = B\,\mathbf{y}$ have a saddle point at the origin.

Consider the ODE $\dot{\mathbf{z}} = C\mathbf{z}$. What different types of fixed point can the origin of this ODE be? Justify your answer. Sketch the different types of phase portraits this system has. [7]

Question 4

(a) Let $A \in \mathbb{R}^{2\times 2}$ be a constant coefficient real matrix with a double eigenvalue λ and only one eigenvector $\mathbf{v}_1$.

(i) Give the definition of a generalised eigenvector of A. [2]

(ii) Show that the general solution of the ODE $\dot{\mathbf{x}} = A\mathbf{x}$ is of the form

$$\mathbf{x}(t) = \left(c_1\,\mathbf{v}_1 + c_2\,(\mathbf{v}_2 + t(A-\lambda I)\mathbf{v}_2)\right)\,e^{\lambda t},$$

and explain what criteria $\mathbf{v}_2$ should satisfy. [5]

(b) Consider the ODE $\dot{\mathbf{x}} = A\mathbf{x}$, with

$$A = \begin{pmatrix} 3 & 0 & 0 \\ -2 & 1 & -2 \\ 7 & 2 & -3 \end{pmatrix}$$

Find the general solution of this ODE. [9]

(c) Consider the system

$$\begin{aligned} \dot{x} &= y - x - 1 \\ \dot{y} &= x - x^2 \end{aligned}$$

(i) Calculate the equilibria of this system and determine their stability. [6]

(ii) Sketch local phase portraits for each equilibrium. [3]

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